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QUANTIZATION IS JUST A CERTAIN REGARD TO...

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Abstract

In signal analysis, the Hilbertian structure associated to a measure set is the mathematical framework for analyzing signals which "live" precisely on the set. A *frame* or *coherent states* quantization consists in selecting a Hilbert subspace which is reproducing. The selection can be motivated either by a statistical reading of experimental data or by the need of focusing on certain aspects of signals. This frame quantization scheme could reveal itself as an efficient tool for quantizing physical systems for which the implementing of more traditional methods is unmanageable. The procedure is first illustrated by the example of infinite- <u>and</u> finite-dimensional quantizations of the particle motion on the line. Interesting

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new inequalities concerning observables emerge from the finite-dimensional quantization, in particular in the context of the quantum Hall effect. We next apply the procedure to the still problematic quantization of the particle motion on the circle. Related to the latter problem is the quantization of dynamics of a test particle in the two-dimensional de Sitter space, the group of symmetry of which is $SO_0(1, 2)$. Our quantization procedure then yields the realization of the corresponding principal series representation of $SO_0(1, 2)$. We also present an application of the method to a toy model for quantum geometry, namely the Ashtekar-Fairhurst-Willis polymer particle representation.

1 Introduction

Besides the fact that mathematics is the language of physics, the organic relation between data and signal processing and physics is widely acknowledged on an epistemological level. Mainly since Galileo, Physics proceeds to an axiomatisation of Nature by converting selected entities identified as objects or phenomena into numbers, or rather into quantifiable data [1]. Actually, an increasingly large part of the common physicist's activity is based on the collection of observational data obtained through more or less elaborate experimental devices and protocoles and selectively considered according to interpretative criteria. One opens a "window" at a certain time of this historical process, because one knows more or less precisely what one wants to search for or to verify, with the "glasses" one has put on and guided by a theoretical/interpretative context (even though some experimental physicists are reluctant to acknowledge this fact). Physics is part of Natural Sciences and its prime object is what we call "Nature", or rather, in a more restrictive sense, "Matter", "Energy", "Interaction", which appears at a certain moment of the process in the form of "significant" data. So the question arises how to process those data, and this arises the question of selected point of view or *frame*. Faced to a set of "raw" collected data encoded into a certain mathematical form and provided by a measure, *i.e.* a function which attributes a weight of importance to subsets of data, we give in addition more or less importance to different aspects of those data by choosing in an opportunistic way the most appropriate frame of analysis.

We would like to include into this general scheme the *quantization processing*, *i.e.* the way of considering objects from a quantum point of view, exactly like we quantize the classical phase space in quantum mechanics. Hence, the aim of the present paper is to advocate the idea that quantization pertains to a larger discipline than just restricting to specific domains of Physics like mechanics or field theory. Our position is even iconoclastic, since we favor a radical change of point of view *vis-à-vis* of Physics : instead of considering data and signal processing just as a tool for experimental/observational physics, we would like to promote that discipline to a more fundamental rank, even to dare to claim that Physics is part of it!

We advance arguments which are based on a mathematical formalism centered on coherent states or frame quantization. In order to become familiar with the mathematics of this approach, we start the body of the paper by giving in Section 2 a comprehensive and pedagogical example : the analysis of the circle considered as an observation set from the point of view of the plane. In Section 3 we really enter the subject by presenting the general mathematical framework, and we apply in Section 4 this formalism to the elementary example of the motion of the particle on the real line. In Section 5 we consider finite-dimensional restrictions of the latter example. Interesting outcomes hold in terms of unexpected inequalities expressing some correlation between respective sizes of the "universe" (or just the sample in which is confined the quantum system, like in the quantum Hall effect), and of the elementary cell explorable by the system. In Section 6 we turn to another simple, but still controversial, example, namely the motion of a particle on the circle. A straightforward adaptation of our results to this case allows to deal with the motion of a test particle in $1+1$ de Sitter space (Section 7). We end our series of examples by the consideration in Section 8 of a problem inspired by modern quantum geometry, where geometric entities are treated as quantum observables, as they have to do in order to be promoted to the status of objects and not to be simply considered as a substantial arena in which physical objects "live". Eventually (Section 9), we shall attempt to give some hints for further research integrating in a systematic way a constitutive duality characterising the probabilistic aspects of the presented material.

2 A 2-dimensional quantization of the unit circle through a continuous frame for the plane

The preliminary example we now present is certainly the simplest one among those we can exhibit in order to become familiar with the formalism of the next sections.

2.1 The interplay between circle (as an observation set) and plane (as a Euclidean space)

Everyone is familiar with the usual orthonormal frame of the Euclidean plane $\mathbb{R}^2$. This frame is defined by two vectors (in Dirac ket notations): $|0\rangle$ and $\left|\frac{\pi}{2}\right\rangle$ such that

$$\langle 0|0\rangle = 1 = \left\langle \frac{\pi}{2}\middle| \frac{\pi}{2}\right\rangle, \ \left\langle 0\middle|\frac{\pi}{2}\right\rangle = 0,$$

and such that the sum of their corresponding orthogonal projectors *solves the identity*

$$\mathbb{I} = |0\rangle\langle 0| + \left|\frac{\pi}{2}\right\rangle\left\langle\frac{\pi}{2}\right|. \tag{1}$$

This is a trivial reinterpretation of the matrix identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Let us now consider the unit vector with polar angle $\theta \in [0, 2\pi)$:

$$|\theta\rangle = \cos\theta|0\rangle + \sin\theta\left|\frac{\pi}{2}\right\rangle. \tag{3}$$

Its corresponding orthogonal projector is given by:

$$|\theta\rangle\langle\theta| = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} = \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}. \tag{4}$$

In an interpretative quantum context, in which the considered system is precisely the plane, the squares of the superposition coefficients in (3), namely $\cos^2\theta$ and $\sin^2\theta$, have a probabilistic meaning: they are the two components of a discrete probability distribution with parameter θ. Borrowing, for a while, to quantum formalism its terminology, the two basic states $|0\rangle$ and $|\frac{\pi}{2}\rangle$ are eigenstates of the quantum observable $\sigma_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$. The latter can be viewed as the orientation of the plane, with two measurable issues : clockwise (-1) or anticlockwise (+1). According to the "Collapse Postulate" and the "Born Rule", carrying out a "measurement" of the observable orientation on the system in the state $|\theta\rangle$ has the effect of collapsing the system into the σ_3-eigenstate $|0\rangle$ (resp. $|\frac{\pi}{2}\rangle$) with probability $|\langle\theta|0\rangle|^2 = \cos^2\theta$ (resp. $|\langle\theta|\frac{\pi}{2}\rangle|^2 = \sin^2\theta$).

Another probabilistic meaning of the quantities $\cos^2\theta$ and $\sin^2\theta$ finds its natural place in the context of *geometrical probability*, as it will be shown in 2.3.

The θ dependent superposition (3) can also be viewed as a *coherent state* superposition in the sense one understands this word in quantum physics, more precisely in quantum optics, the field in which it was introduced by Glauber [3] (see also [2]). One of the most salient features of coherent states lies in the fact that they solve the identity in the vector space they belong to. Indeed, integrating the matrix elements of (4) over all angles and dividing by π leads to a *continuous* analogue of (1)

$$\frac{1}{\pi}\int_0^{2\pi} d\theta|\theta\rangle\langle\theta| = \mathbb{I}. \tag{5}$$

We thus have obtained a continuous frame for the plane, that is to say the continuous set of unit vectors forming the unit circle, for describing, with an extreme redundancy, the euclidean plane. The operator relation (5) is equally understood through its action on a vector $|v\rangle = \|v\|\,|\phi\rangle$ with polar coordinates $\|v\|, \phi$. By virtue of $\langle\theta|\theta'\rangle = \cos(\theta - \theta')$ we have :

$$|v\rangle = \frac{\|v\|}{\pi}\int_0^{2\pi} d\theta\, \cos(\phi - \theta)|\theta\rangle, \tag{6}$$

a relation which illustrates the *overcompleteness* of the family $\{|\theta\rangle\}$. The vectors of this family are not linearly independent, and their mutual "overlappings" are given by the scalar products $\langle\phi|\theta\rangle = \cos(\phi - \theta)$.

Another way to understand the above continuous frame is to consider the isometric embedding of the euclidean plane into the real Hilbert space of trigonometric Fourier series, *i.e.* the Hilbert space $L^2_{\mathbb{R}}(S^1, \frac{d\theta}{\pi}) \equiv L^2(S^1)$ of real-valued square-integrable functions on the circle, endowed with the scalar product

$$\langle f|g\rangle_{L^2} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)g(\theta)\, d\theta. \tag{7}$$

Let $|v\rangle$ be a vector with polar coordinates r, ϕ. There corresponds to $|v\rangle$ the function

$$v(\theta) = \langle\theta|v\rangle = r\cos(\phi - \theta), \tag{8}$$

and this is clearly an element of $L^2(S^1)$. We have an isometry since a straightforward application of the resolution of the unity (5) leads to:

$$\langle v_1|v_2\rangle_{L^2} = \frac{1}{\pi}\int_0^{2\pi} v_1(\theta)v_2(\theta)\,d\theta = \frac{1}{\pi}\int_0^{2\pi} \langle v_1|\theta\rangle\langle\theta|v_2\rangle\,d\theta = \langle v_1|v_2\rangle. \tag{9}$$

Also, an interesting outcome of (5) and the consequent embedding (8) is the fact that we can consider the Euclidean plane as a reproducing Hilbert space as well, with precisely its kernel equal to

$$\langle\theta|\theta'\rangle = \cos(\theta-\theta'). \tag{10}$$

Indeed, combining (6) with (8) leads to the integral identity for any 2-dimensional vector $|v\rangle$:

$$v(\theta) = \langle\theta|v\rangle = \frac{1}{\pi}\int_0^{2\pi} \langle\theta|\theta'\rangle v(\theta')\,d\theta. \tag{11}$$

At this point we can adopt another point of view: starting from the circle S^1 provided with the measure $\frac{d\theta}{\pi}$, we consider all possible real-valued square-integrable functions forming $L^2(S^1)$. Then we choose the orthonormal system composed of the two fundamental *harmonics*, $\cos\theta$ and $\sin\theta$. Note that the normalization gives also rise to a probabilistic interpretation since then the functions $\frac{1}{\pi}\cos^2\theta$ and $\frac{1}{\pi}\sin^2\theta$ are two continuous probability distributions of the Wigner type. We build the superposition (3) and we obtain the resolution (5) which can also be viewed as the orthogonal projector mapping $L^2(S^1)$ onto the reproducing two-dimensional subspace spanned by $\cos\theta$ and $\sin\theta$ identified to $|0\rangle$ and $|\frac{\pi}{2}\rangle$ respectively.

2.2 Quantization and symbol calculus

The existence of the continuous frame $\{|\theta\rangle\}$ offers the possibility to proceed to a two-dimensional *quantization* of the circle. More precisely, by quantization of a function or *classical* observable $f(\theta)$ on the circle, we mean the linear application defined by

$$f \mapsto \mathcal{O}_f = \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\theta)|\theta\rangle\langle\theta| \tag{12}$$

which associates to f the linear operator $\mathcal{O}_f$ in the plane.

For instance, let us choose the angle function $f(\theta) = \theta$. Its quantized version is equal to the matrix :

$$\mathcal{O}_\theta = \begin{pmatrix} \pi & -\frac{1}{2} \\ -\frac{1}{2} & \pi \end{pmatrix}, \tag{13}$$

with eigenvalues $\pi \pm \frac{1}{2}$.

The frame $\{|\theta\rangle\}$ allows one to carry out a symbol calculus *à la* Berezin-Lieb [4, 5]. This means that to any *self-adjoint* linear operator $\mathcal{O}$, *i.e.* a real symmetric matrix with respect to some orthonormal basis, one can associate two types of *symbol*, functions $\check{\mathcal{O}}(\theta)$ et $\hat{\mathcal{O}}(\theta)$ respectively defined on the unit circle by

$$\check{\mathcal{O}}(\theta) = \langle\theta|\mathcal{O}|\theta\rangle \ : \ \textit{lower or covariant symbol}, \tag{14}$$

which is precisely the mean value of the quantum observable $\mathcal{O}$ in the state $|\theta\rangle$, and

$$\mathcal{O} = \frac{1}{\pi}\int_0^{2\pi} d\theta\, \hat{\mathcal{O}}(\theta)|\theta\rangle\langle\theta|. \tag{15}$$

The function $\hat{\mathcal{O}}(\theta)$ which appears in this operator-valued integral is called *upper or contravariant symbol*. It is highly non-unique, but will be chosen as the simplest one.

Three basic matrices generate the Jordan algebra of all real symmetric 2×2 matrices. They are the identity matrix, the symbol of which is trivially the function 1, and the two real Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

Any element $\mathcal{O}$ of the algebra decomposes as :

$$\mathcal{O} \equiv \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \frac{a+d}{2}\mathbb{I} + \frac{a-d}{2}\sigma_3 + b\sigma_1 \equiv \alpha\mathbb{I} + \delta\sigma_3 + \beta\sigma_1. \tag{17}$$

The product in this algebra is defined by

$$\mathcal{O}'' = \mathcal{O} \odot \mathcal{O}' = \frac{1}{2}\left(\mathcal{O}\mathcal{O}' + \mathcal{O}'\mathcal{O}\right), \tag{18}$$

which entails on the level of components α, δ, β, the relations :

$$\alpha'' = \alpha\alpha' + \delta\delta' + \beta\beta', \ \delta'' = \alpha\delta' + \alpha'\delta, \ \beta'' = \alpha\beta' + \alpha'\beta. \tag{19}$$

The upper and lower symbols of non-trivial basic elements are respectively given by :

$$\cos 2\theta = \check{\sigma}_3(\theta) = \frac{1}{2}\hat{\sigma}_3(\theta), \; \sin 2\theta = \check{\sigma}_1(\theta) = \frac{1}{2}\hat{\sigma}_1(\theta). \tag{20}$$

There follows for the symmetric matrix (17) the two symbols :

$$\begin{aligned} \check{\mathcal{O}}(\theta) &= \frac{a+d}{2} + \frac{a-d}{2}\cos 2\theta + b\sin 2\theta = \alpha + \delta\cos 2\theta + \beta \sin 2\theta, \\ \hat{\mathcal{O}}(\theta) &= \alpha + 2\delta\cos 2\theta + 2\beta\sin 2\theta = 2\check{\mathcal{O}}(\theta) - \frac{1}{2}\mathrm{Tr}\mathcal{O}. \end{aligned} \tag{21}$$

For instance, the lower symbol of the "quantum angle" (13) is equal to

$$\langle \theta | \mathcal{O}_\theta | \theta \rangle = \pi - \frac{1}{2}\sin 2\theta, \tag{22}$$

a π-periodic function which smoothly varies between the two eigenvalues of $\mathcal{O}_\theta$. One should notice that all these symbols belong to the subspace $\mathcal{V}_\mathcal{O}$ of real Fourier series which is the closure of the linear span of the three functions 1, $\cos 2\theta$, $\sin 2\theta$. Note that the subspace of symbols is the closure of the linear span of the algebraic square of the Euclidean plane viewed a subset of $L^2(S^1)$. Also note that $\hat{\mathcal{O}}(\theta)$ is defined up to the addition of any function $\mathcal{N}(\theta)$ which makes (15) vanish. Such a function lives in the orthogonal complement of $\mathcal{V}_\mathcal{O}$.

The Jordan multiplication law (18) is commutative but not associative and its counterpart on the level of symbols is the so-called $\star$-product. For instance, we have for the upper symbols :

$$\widehat{\mathcal{O}\odot\mathcal{O}'}(\theta) \equiv \hat{\mathcal{O}}(\theta)\star\hat{\mathcal{O}}'(\theta) = \alpha\hat{\mathcal{O}}'(\theta) + \alpha'\hat{\mathcal{O}}(\theta) + \delta\delta' + \beta\beta' - \alpha\alpha', \tag{23}$$

and the formula for lower symbols is the same.

That terminology of *lower/upper* is justified by two inequalities, named Berezin-Lieb inequalities, which follow from the symbol formalism. Let g be a convex function. Denoting by $\lambda_\pm$ the eigenvalues of the symmetric matrix $\mathcal{O}$, we have

$$\frac{1}{\pi}\int_0^{2\pi} g(\check{\mathcal{O}}(\theta))\, d\theta \leq \mathrm{Tr} g(\mathcal{O}) = g(\lambda_+) + g(\lambda_-) \leq \frac{1}{\pi}\int_0^{2\pi} g(\hat{\mathcal{O}}(\theta))\, d\theta. \tag{24}$$

This double inequality is not trivial. Independently of the Euclidean context it reads as :

$$\langle g(t + r\cos\theta)\rangle \leq \frac{1}{2}\left[g(t+r) + g(t-r)\right] \leq \langle g(t + 2r\cos\theta)\rangle, \quad (25)$$

where $t \in \mathbb{R}, r \geq 0$ and $\langle \cdot \rangle$ denotes the mean value on a period. If one applies (25) to the exponential function $g(X) = e^X$, we get an intertwining of inequalities involving Bessel functions of the second kind and the hyperbolic cosine :

$$\cdots \leq I_0(x) \leq \cosh x \leq I_0(2x) \leq \cosh 2x \leq \cdots \quad \forall x \in \mathbb{R}. \quad (26)$$

2.3 Probabilistic aspects

Behind the resolution of the identity (5) lies an interesting interpretation in terms of geometrical probability. Let us consider a Borel subset Δ of the interval $[0, 2\pi)$ and the restriction to Δ of the integral (5) :

$$a(\Delta) = \frac{1}{\pi}\int_{\Delta} d\theta |\theta\rangle\langle\theta|. \quad (27)$$

One easily verifies the following properties :

$$\begin{aligned} &a(\emptyset) = 0, \; a([0, 2\pi)) = \mathbb{I}, \\ &a(\cup_{i\in J}\Delta_i) = \sum_{i\in J} a(\Delta_i), \; \Delta_i \cap \Delta_j = \emptyset \text{ if for all } i \neq j. \end{aligned} \quad (28)$$

The application $\Delta \mapsto a(\Delta)$ defines a normalized measure on the σ-algebra of the Borel sets in the interval $[0, 2\pi)$, assuming its values in the set of positive linear operators on the Euclidean plane (POV measure). Denoting the measure density $(1/\pi)|\theta\rangle\langle\theta|\, d\theta$ by $a(d\theta)$ we shall also write

$$a(\Delta) = \int_{\Delta} a(d\theta). \quad (29)$$

Let us now put into evidence the probabilistic nature of the measure $a(\Delta)$. Let $|\phi\rangle$ be a unit vector. The application

$$\Delta \mapsto \langle\phi|a(\Delta)|\phi\rangle = \frac{1}{\pi}\int_{\Delta} \cos^2(\theta - \phi)\, d\theta \quad (30)$$

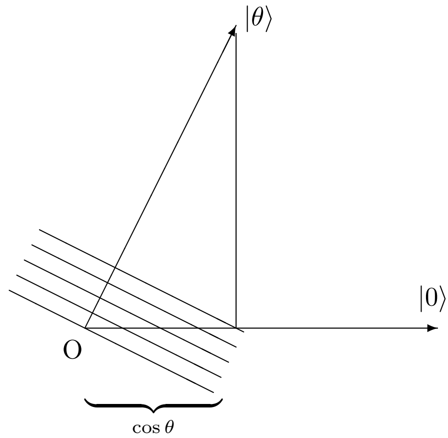

Figure 1: Set $\{D_{\theta,p}\}$ of straight lines normal to $|\theta\rangle$ which intersect the segment with origin O and length $|\cos\theta|$ equal to the projection of $|\theta\rangle$ onto $|0\rangle$.

is clearly a probability measure. It is positive, of total mass 1, and it inherits σ-additivity from $a(\Delta)$. Now, the quantity $\langle\phi|a(\Delta)|\phi\rangle$ means that direction $|\phi\rangle$ is examined from the point of view of the family of vectors $\{|\theta\rangle,\ \theta\in\Delta\}$. As a matter of fact, it has a *geometrical probability* interpretation in the plane [6]. With no loss of generality let us choose $\phi = 0$. Recall here the canonical equation describing a straight line $D_{\theta,p}$ in the plane :

$$\langle\theta|u\rangle \equiv \cos\theta\, x + \sin\theta\, y = p, \tag{31}$$

where $|\theta\,\rangle$ is the direction normal to $D_{\theta,p}$ and the parameter p is equal to the distance of $D_{\theta,p}$ to the origin. There results that $dp\, d\theta$ is the (non-normalized) probability measure element on the set $\{D_{\theta,p}\}$ of the lines randomly chosen in the plane. Picking a certain θ, consider the set $\{D_{\theta,p}\}$ of the lines normal to $|\theta\rangle$ which intersect the segment with origin O and length $|\cos\theta|$ equal to the projection of $|\theta\rangle$ onto $|0\rangle$ as shown in Figure 1

The measure of this set is equal to :

$$\left(\int_0^{\cos^2\theta} dp\right) d\theta = \cos^2\theta\, d\theta. \tag{32}$$

Integrating (32) on all directions $|\theta\rangle$ gives the area of the unit circle. Hence we can construe $\langle\phi|a(\Delta)|\phi\rangle$ as the probability for a straight line in the plane to belong to the set of secants of segments which are projections $\langle\phi|\theta\rangle$ of the unit vectors $|\theta\rangle$, $\theta \in \Delta$ onto the unit vector $|\phi\rangle$. One could think in terms of *polarizer* $\langle\theta|$ and *analyzer* $|\theta\rangle$ "sandwiching" the directional signal $|\phi\rangle$.

2.4 Remark : A 2-dimensional quantization of the interval $[0,\pi)$ through a continuous frame for the half-plane

From a strictly quantal point of view, we should have made equivalent any vector $|v\rangle$ of the Euclidean plane with its symmetric $-|v\rangle$, which amounts to deal with the half-plane viewed as the coset $\mathbb{R}^2/\mathbb{Z}_2$. Following the same procedure as above, we start from the Hilbert space $L^2\left([0,\pi), \frac{2}{\pi}\, d\theta\right)$ and we choose the <u>same</u> subset $\{\cos\theta, \sin\theta\}$ which is <u>still</u> orthonormal. Note that $\frac{2}{\pi}\cos^2\theta$ and $\frac{2}{\pi}\sin^2\theta$ are now exactly Wigner semicircle distributions. We then consider the continuous family (3) of coherent states $|\theta\rangle$. They are normalized and they solve the identity exactly like in (5) (just change the factor $\frac{1}{\pi}$ into $\frac{2}{\pi}$). The previous material can be repeated *in extenso* but with the restriction that $\theta \in [0,\pi)$.

3 Coherent state quantization

We now have reached the point at which we can think about a general formalism underlying the above coherent state construction and quantization procedure. The approach we are going to follow from now on is mainly based on Refs.[7, 8].

Quantum Mechanics and Signal Analysis have many aspects in common. As a departure point of their respective formalism, one finds a *raw* set X of basic parameters or data that we denote by x. This set may be a classical phase space in the former case, like the complex plane for the particle motion on the line, whereas it might be a temporal line or a time-frequency half-plane in the latter one. Actually it can be any set of data accessible to observation, for instance the circle or some interval like in the previous section, and the minimal significant structure one requires so far is the existence of a measure $\mu(dx)$ on X. As a measure space, X will be given the name of an *observation* set, and the existence of a measure provides us with a statistical reading of the set of all measurable real or complex valued functions $f(x)$ on X: it allows us to compute for instance average values on subsets with bounded measure. Actually, both theories deal with quadratic mean values, and the natural framework of study is the Hilbert space $L^2(X, \mu)$ of all square-integrable functions $f(x)$ on the observation set X: $\int_X |f(x)|^2 \, \mu(dx) < \infty$. The function f is referred to as *finite-energy* signal in Signal Analysis and as (pure) quantum state in Quantum Mechanics. However, it is precisely at this stage that "quantum processing" of X differs from signal processing in at least three points:

1. not all square-integrable functions are eligible as quantum states,
2. a quantum state is defined up to a nonzero factor,
3. among the functions $f(x)$, those that are eligible as quantum states and that are of unit norm, $\int_X |f(x)|^2 \, \mu(dx) = 1$, give rise to a probabilistic interpretation: the correspondence $X \supset \Delta \mapsto \int_\Delta |f(x)|^2 \mu(dx)$ is a probability measure which is interpreted in terms of localization in the measurable set Δ and which allows to determine mean values of quantum observables, (essentially) self-adjoint operators defined in a domain that is included in the set of quantum states.

The first point lies at the heart of the *quantization* problem: what is the more or less canonical procedure allowing to select quantum states among simple signals? In other words, how to select the true (projective) Hilbert space of quantum states, denoted by $\mathcal{H}$, *i.e.* a closed subspace of $L^2(X, \mu)$, or equivalently the corresponding orthogonal projector $\mathbb{I}_{\mathcal{H}}$?

This problem can be solved if one finds a map from X to $\mathcal{H}$, $x \mapsto |x\rangle \in \mathcal{H}$ (in Dirac notation), defining a family of states $\{|x\rangle\}_{x \in X}$ obeying the following two conditions:

- **normalization**

$$\langle\, x \,|x\rangle = 1, \tag{33}$$

- **resolution of the unity in $\mathcal{H}$**

$$\int_X |x\rangle\langle x|\,\nu(dx) = \mathbb{I}_{\mathcal{H}}, \tag{34}$$

  where $\nu(dx)$ is another measure on X, usually absolutely continuous with respect to $\mu(dx)$: this means that there exists a positive measurable function $h(x)$ such that $\nu(dx) = h(x)\mu(dx)$.

The quantization of a *classical* observable, that is to say of a function $f(x)$ on X having specific properties with respect to some supplementary structure allocated to X, like topology, geometry or something else, simply consists in associating to the function $f(x)$ the operator defined by

$$A_f = \int_X f(x)|x\rangle\langle x|\,\nu(dx). \tag{35}$$

Note that the map $f \mapsto A_f$ is linear and that the function $f(x) = 1$ goes to the identity operator. In the present context like in Section 2, the function $f(x) \equiv \widehat{A}_f(x)$ is named upper symbol of the operator A_f by Lieb [5] (or contravariant by Berezin [4]), whereas the mean value $\langle x|A_f|x\rangle \equiv \check{A}_f(x)$ is the lower (or covariant) symbol of A_f. One can say that, according to this approach, a quantization of the observation set is in one-to-one correspondence with the choice of a frame in the sense of (33) and (34). Actually, the term of *frame* [10] is more appropriate for designating the total family $\{|x\rangle\}_{x\in X}$. To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with X. This frame can be discrete or continuous, depending on the topology furthermore allocated to the set X, and it can be overcomplete, of course. The validity of a precise frame choice is determined by comparing spectral characteristics of quantum observables A_f with experimental data. Of course, such a particular quantization scheme, associated to a specific frame, is intrinsically limited to all those classical observables for which the expansion (35) is mathematically justified within the theory of operators in Hilbert space (*e.g.* weak convergence). However, it is well known that limitations hold for *any* quantization scheme.

The explicit construction of a frame as well as its physical relevance are clearly crucial. It is remarkable that signal and quantum formalisms meet again on this level, since the frame is called, in a wide sense, a *wavelet* family [9] or a *coherent state* family [11] according to the practitioner's filiation. Two methods for constructing such families are generally in use. The first one rests upon group representation theory: a specific state or *probe*, say $|x_0\rangle$, is transported along the orbit $\{|g \cdot x_0 \equiv x\rangle\}_{g\in G}$ by the action of a group G for which X is a homogeneous space. Irreducibility (Schur's Lemma) and unitarity conditions, combined with square integrability of the representation in some restricted sense, automatically lead to properties (33) and (34). Various examples of such group-theoretical constructions are given in [12, 11]. The second method has a wave packet flavor in the sense that the state $|x\rangle$ is obtained from some superposition of elements in a fixed family of states $\{|\lambda\rangle\}_{\lambda\in\Lambda}$ which is total in $\mathcal{H}$:

$$|x\rangle = \int_\Lambda |\lambda\rangle\, \sigma(x, d\lambda). \tag{36}$$

Here, the complex-valued x-dependent measure σ has its support Λ contained in the support of the spectral resolution $E(d\lambda)$ of a certain self-adjoint operator A, and the $|\lambda\,\rangle$'s are precisely eigenstates of A: $A|\lambda\,\rangle = \lambda|\lambda\,\rangle$. More precisely, they can be eigenstates in a distributional sense so as to put into the game of the construction portions belonging to the possible continuous part of the spectrum of A. Examples of such wave-packet constructions are given in [10, 13, 14, 15, 16], and we shall follow a similar procedure in the present paper.

For pedagogical purposes, we now suppose that A has a only discrete spectrum, say $\{a_n, 0 \leq n \leq N\}$, with N finite or infinite. Normalized eigenstates are denoted by $|n\rangle$ (Fock notation) and they form an orthonormal basis of $\mathcal{H}$. Now, suppose that the basis $\{|n\rangle\}_{0\leq n\leq N}$ is in one-to-one correspondence with an orthonormal set $\{\phi_n(x)\}_{0\leq n\leq N}$ of elements of $L^2(X, \mu)$. Furthermore, and this a decisive step in the wave packet construction, we assume that

$$0 < \mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere on } X, \tag{37}$$

and the above Hilbertian superposition makes sense provided that set X be equipped of a mild topological structure for which this map is continuous. Then, the states

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \phi_n^*(x)|n\rangle, \tag{38}$$

satisfy both of our requirements (33) and (34). Indeed, the normalization is automatically ensured because of the orthonormality of the set $\{|n\rangle\}$ and the presence of the normalization factor (37). The resolution of the unity in $\mathcal{H}$ holds by virtue of the orthonormality of the set $\{\phi_n(x)\}$ if $\nu(dx)$ is related to $\mu(dx)$ by

$$\nu(dx) = \mathcal{N}(x)\mu(dx). \tag{39}$$

The resolution of the unity in $\mathcal{H}$ can alternatively be understood in terms of the scalar product $\langle\, x\,|x'\rangle$ of two states of the family. Indeed, (34) implies that, to any vector $|\phi\rangle$ in $\mathcal{H}$, one can isometrically associate the function

$$\phi(x) \equiv \sqrt{\mathcal{N}(x)}\langle x\,|\phi\rangle \tag{40}$$

in $L^2(X,\mu)$, and this function obeys

$$\phi(x) = \int_X \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x|x'\rangle\phi(x')\,\mu(dx'). \tag{41}$$

Hence, $\mathcal{H}$ is isometric to a reproducing Hilbert space with kernel

$$\mathcal{K}(x,x') = \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x\,|x'\rangle, \tag{42}$$

and the latter assumes finite diagonal values ($a.e.$), $\mathcal{K}(x,x) = \mathcal{N}(x)$, by construction.

A last point of this quantization scheme concerns its statistical aspects, already pointed out in Section 2. There is indeed an interplay between two probability distributions :

- for each n, a "continuous" distribution on (X,μ),

$$X \ni x \mapsto |\phi_n(x)|^2, \tag{43}$$

- for almost each x, a discrete distribution,

$$n \mapsto \frac{|\phi_n(x)|^2}{\mathcal{N}(x)}. \tag{44}$$

Hence, a probabilistic approach to experimental observations should serve of guideline in choosing the set of the $\phi_n(x)$'s.

4 A standard example: quantization of the motion of a particle on the line

We now enter traditional Quantum Physics by considering one of its pedagogical models, namely the quantum version of the particle motion on the real line. On the classical level, the corresponding phase space is $X = \mathbb{R}^2 \simeq \mathbb{C} = \{z = \frac{1}{\sqrt{2}}(q + ip)\}$ (in complex notation and with suitable physical units). Let us provide it with the ordinary Lebesgue measure on the plane which coincides here with the symplectic 2-form : $\mu(dz\, dz^*) \equiv \frac{1}{\pi}\, d^2z$ where $d^2z = d\Re z\, d\Im z$. Strictly included in the Hilbert space $L^2(\mathbb{C}, \mu(dz\, dz^*))$ of all complex-valued functions on the complex plane which are square-integrable with respect to this measure, there is the so-called *Fock-Bargmann Hilbert* subspace $\mathcal{H}$ of all square integrable functions which are of the form $\phi(z) = e^{-\frac{|z|^2}{2}} g(z^*)$ where $g(z^*)$ is anti-analytical entire. An obvious orthonormal basis of this subspace $\mathcal{H}$ is formed of the normalized powers of the conjugate of the complex variable z weighted by the Gaussian , *i.e.* $\phi_n(z) \equiv e^{-\frac{|z|^2}{2}} \dfrac{z^{*n}}{\sqrt{n!}}$ with $n \in \mathbb{N}$. Since $\mathcal{N}(z) = \sum_{n=0}^{+\infty} |\phi_n(z)|^2 = 1$, we can consider the following infinite linear superposition in $\mathcal{H}$

$$|z\rangle = \frac{1}{\sqrt{\mathcal{N}(z)}} \sum_n \phi_n^*(z)|n\rangle = e^{-\frac{|z|^2}{2}} \sum_{n\in\mathbb{N}} \frac{z^n}{\sqrt{n!}}|n\rangle, \tag{45}$$

The Fock notation $\{|n\rangle \equiv \phi_n\}_{n\in\mathbb{N}}$ is not fortuitous here because these states, as eigenstates of the Euler or *number operator*, $N = z^* \dfrac{\partial}{\partial z^*} + \dfrac{|z|^2}{2}$, are nothing but the eigenstates of the harmonic oscillator in the Fock-Bargmann representation. From the general construction of Section 3 one gets the two fundamental features of the states (45), namely normalization and unity resolution:

$$\langle z\,|z\rangle = 1, \quad \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\, d^2z = \mathbb{I}_{\mathcal{H}}. \tag{46}$$

We should notice here the probabilistic features of the orthonormal set $\{\phi_n(z),\, n \in \mathbb{N}\}$: the discrete distribution $n \mapsto \dfrac{|\phi_n(z)|^2}{\mathcal{N}(z)} =$

$e^{-|z|^2}\dfrac{|z|^{2n}}{n!}$ is a Poisson distribution with average number of occurrences equal to $|z|^2$, and the continuous distribution $z \mapsto |\phi_n(z)|^2 = e^{-|z|^2}\dfrac{|z|^{2n}}{n!}$ is a Gamma distribution (with respect to the square of the radial variable) with n as a shape parameter.

The states (45) are the well-known Klauder-Glauber-Sudarshan coherent states [3, 17], actually discovered in the early days of Quantum Mechanics by Schrödinger [18] and christened *coherent* by Glauber within the context of quantum optics (where $|z|^2$ has to be interpreted as the expected number of photons). Among numerous interesting properties enjoyed by them, a fundamental one is their quality of being *canonical quantizers* [4]. From Section 3 we already know that any classical observable f, that is a (usually supposed smooth) function of phase space variables (q, p) or equivalently of (z, z^*), is transformed through the operator integral

$$\frac{1}{\pi}\int_{\mathbb{C}} f(z, z^*)\, |z\rangle\langle z|\, d^2 z = A_f, \tag{47}$$

into an operator A_f acting on the Hilbert space $\mathcal{H}$ of quantum states. Now, we have for the most basic one,

$$\frac{1}{\pi}\int_{\mathbb{C}} z\, |z\rangle\langle z|\, d^2 z = \sum_n \sqrt{n+1}|n\rangle\langle n+1| \equiv a, \tag{48}$$

which is the lowering operator, $a|n\rangle = \sqrt{n}|n-1\rangle$. We easily check that the coherent states are eigenvectors of $a :\ a|z\rangle = z|z\rangle$. The adjoint $a^\dagger$ is obtained by replacing z by z^* in (48), and we get the factorisation $N = a^\dagger a$ for the number operator, $N|n\rangle = n|n\rangle$, together with the commutation rule $[a, a^\dagger] = \mathbb{I}_{\mathcal{H}}$. The lower symbol or expected value of the number operator $\langle z|N|z\rangle$ is precisely $|z|^2$.

From $q = \frac{1}{\sqrt{2}}(z + z^*)$ et $p = \frac{1}{\sqrt{2}i}(z - z^*)$, one easily infers by linearity that the canonical position q and momentum p map to the quantum observables $\frac{1}{\sqrt{2}}(a + a^\dagger) \equiv Q$ and $\frac{1}{\sqrt{2}i}(a - a^\dagger) \equiv P$ respectively. In consequence, the self-adjoint operators Q and P obey the canonical commutation rule $[Q, P] = i\mathbb{I}_{\mathcal{H}}$, and for this reason fully deserve the name of position and momentum operators of the usual (galilean) quantum mechanics, together with all localization properties specific to the latter. In this context, it is worthy to recall what *quantization of classical mechanics* does mean in a commonly accepted sense (for a recent and complete review see [19]). In the above we have chosen units such that the Planck constant is just put equal to 1. Here we reintroduce it since it parametrizes the link between classical and quantum mechanics.

Van Hove canonical quantization rules [20]

Given a phase space with canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$

- to the classical observable $f(\boldsymbol{q}, \boldsymbol{p}) = 1$ corresponds the identity operator in the (projective) Hilbert space $\mathcal{H}$ of quantum states,
- the correspondence that assigns to a classical observable $f(\boldsymbol{q}, \boldsymbol{p})$ a (essentially) self-adjoint operator on $\mathcal{H}$ is a linear map,
- to the classical Poisson bracket corresponds, <u>at least at the order $\hbar$</u>, the quantum commutator, multiplied by $i\hbar$:

$$\text{with } f_j(\boldsymbol{q}, \boldsymbol{p}) \mapsto A_{f_j} \text{ for } j = 1, 2, 3$$
$$\text{we have } \{f_1, f_2\} = f_3 \mapsto [A_{f_1}, A_{f_2}] = i\hbar A_{f_3} + o(\hbar),$$

- some conditions of minimality on the resulting observable algebra.

The last point can give rise to technical and interpretational difficulties.

5 Finite-dimensional canonical case

The idea of exploring various aspects of Quantum Mechanics by restricting the Hilbertian framework to finite-dimensional space is not new, and has been intensively used in the last decade, mainly in the context of Quantum Optics [21, 22], but also in the perpective of non-commutative geometry and "fuzzy" geometric objects [23], or in matrix model approaches in problems like the quantum Hall effect [24]. For Quantum Optics, a comprehensive review (mainly devoted to the Wigner function) is provided by Ref. [25]. In [22], the authors defined normalized finite-dimensional coherent states by truncating the Fock expansion of the standard coherent states. Let us see through the approach presented in Section 2 how we recover [26] their coherent states. We just restrict the choice of the orthonormal set $\{\phi_n\}$ to a finite subset of it, more precisely to the N first elements.

$$\phi_n(z) = e^{-\frac{|z|^2}{2}} \frac{z^{*n}}{\sqrt{n!}},\ n = 0, 1, \dots N-1 < \infty. \tag{49}$$

The coherent states then read :

$$|z\rangle = \frac{e^{-\frac{|z|^2}{2}}}{\sqrt{\mathcal{N}(z)}} \sum_{n=0}^{N-1} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{50}$$

with

$$\mathcal{N}(z) = e^{-|z|^2} \sum_{n=0}^{N-1} \frac{|z|^{2n}}{n!}. \tag{51}$$

Following the same quantization procedure yields for the classical observables q and p quantum position and momentum (or "quadratures" in Quantum Optics"). They are now the $N \times N$ matrices,

$$Q_N = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 & \sqrt{\frac{N-1}{2}} \\ 0 & 0 & \dots & \sqrt{\frac{N-1}{2}} & 0 \end{pmatrix}, \tag{52}$$

$$P_N = -i \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 1 & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 & \sqrt{\frac{N-1}{2}} \\ 0 & 0 & \dots & -\sqrt{\frac{N-1}{2}} & 0 \end{pmatrix}. \tag{53}$$

Their commutator is "almost" canonical:

$$[Q_N, P_N] = iI_N - iNE_N, \tag{54}$$

where E_N is the orthogonal projector on the last basis element,

$$E_N = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} .$$

The appearing of such a projector in (54) is clearly a consequence of the truncation at the N^{th} level.

A very interesting issue comes out from this finite-dimensional quantization of the classical phase space. Let us examine the spectral values of the position operator, *i.e.* the allowed or experimentally measurable quantum positions. They are just the zeros of the Hermite polynomials $\mathrm{H}_N(\lambda)$, and the same result holds for the momentum operator. We know that $\mathrm{H}_N(0) = 0$ if and only if N is odd and that the other zeros form a set symmetrical with respect to the origin. Let us order the non-null roots of the Hermite polynomial $\mathrm{H}_N(\lambda)$ as :

$$\left\{-\lambda_{\lfloor \frac{N}{2} \rfloor}(N), -\lambda_{\lfloor \frac{N}{2} \rfloor -1}(N), \dots, -\lambda_1(N), \lambda_1(N), \dots, \lambda_{\lfloor \frac{N}{2} \rfloor -1}(N), \lambda_{\lfloor \frac{N}{2} \rfloor}(N)\right\}, \tag{55}$$

It is a well-known property of the Hermite polynomials that $\lambda_{i+1}(N) - \lambda_i(N) > \lambda_1(N)$ for all $i \geq 1$ if N is odd, whereas $\lambda_{i+1}(N) - \lambda_i(N) > 2\lambda_1(N)$ for all $i \geq 1$ if N is even, and that the zeros of the Hermite polynomials H_N and H_{N+1} intertwine.

In [26], we have studied numerically the product

$$\varpi_N = \lambda_m(N)\lambda_M(N) \tag{56}$$

where $\lambda_M(N) = \lambda_{\lfloor \frac{N}{2} \rfloor}(N)$ (resp. $\lambda_m(N) = \lambda_1(N)$) is the largest root (resp. smallest nonzero root in absolute value) of H_N. We have found that ϖ_N goes asymptotically to π for large even N and to 2π for large odd N.

This result (which can be rigorously proved by using the Wigner semi-circle law for the asymptotic distribution of zeros of the Hermite polynomials [27]) could reveal itself important with regard to its physical implications in terms of correlation between small and large distances. Define by

- $\Delta_N(Q) = 2\lambda_M(N)$ the "size" of the "universe" accessible to exploration by the quantum system,

N	$\delta_N(Q)\Delta_N(Q)$	2π
10	4.713054	
55	5.774856	
100	5.941534	
551	6.173778	
1 000	6.209670	
5 555	6.259760	
10 000	6.267356	
55 255	6.278122	
100 000	6.279776	
500 555	6.282020	
1 000 000	6.282450	6.2831853

Table 1: Values of $\sigma_N = \delta_N(Q)\Delta_N(Q)$ up to $N = 10^6$. Compare with the value of 2π.

- $\delta_N(Q) = \lambda_m(N)$ (resp. $\delta_N(Q) = 2\lambda_m(N)$) for odd (resp. even) N, the "size" of the smallest "cell" forbidden to exploration by the same system,

then σ_N, as a function of N, is strictly increasing and goes asymptotically to 2π :

Hence, we can assert the new inequalities concerning the quantum position and momentum:

$$\delta_N(Q)\Delta_N(Q) \leq 2\pi,\ \delta_N(P)\Delta_N(P) \leq\ 2\pi\ \forall N. \tag{57}$$

In order to fully perceive the physical meaning of such inequalities, it is necessary to reintegrate into them physical constants or scales proper to the considered physical system, *i.e.* characteristic length l_c and momentum p_c.

$$\delta_N(Q)\Delta_N(Q) \leq 2\pi l_c^2,\ \delta_N(P)\Delta_N(P) \leq 2\pi p_c^2\ \forall N, \tag{58}$$

where $\delta_N(Q)$ and $\Delta_N(Q)$ are now expressed in unit l_c.

Let us comment this result from a physical point of view. Realistically, in any physical situation, N cannot be infinite: there is an obvious limitation on frequencies or energies accessible to observation/experimentation. So it is natural to work with a finite although large value of N, which need not be determinate. In consequence, there exists irreducible limitations, namely $\delta_N(Q)$ and $\Delta_N(Q)$ in the exploration of small and large distances, and both limitations have the correlation $\delta_N(Q)\Delta_N(Q) \leq 2\pi l_c^2$.

Suppose there exists, for theoretical reasons, a fundamental or "universal" minimal length, say l_m, something like the Planck length, or equivalently an universal ratio $\rho_u = l_c/l_m \geq 1$. Then, from $\delta_N(Q) \geq l_m$ we infer that there exists a universal maximal length l_M given by

$$l_M \approx 2\pi\rho_u l_c. \tag{59}$$

Of course, if we choose $l_m = l_c$, then the size of the "universe" is $l_M \approx 2\pi l_m$. Now, if we choose a characteristic length proper to Atomic Physics, like the Bohr radius, $l_c \approx 10^{-10}$m, and for the minimal length the Planck length, $l_m \approx 10^{-35}$m, we find for the maximal size the astronomical quantity $l_M \approx 10^{16}$m. On the other hand, if we consider the (controversial) estimate size of our present universe $L_u = cT_u$, with $T_u \approx 13\,10^9$ years, we get from $l_p\, L_u \approx 2\pi l_c^2$ a characteristic length $l_c \approx 10^{-5}$m, *i.e.* a wavelength in the infrared electromagnetic spectrum...

Of course, we should be very cautious about drawing sound physical consequences from the existence of the inequalities (58). Indeed, one can argue that our scheme of quantization leading to such inequalities is strongly dependent on the choice of orthonormal states used in constructing the "quantizer" frame. The physical interpretation of the inequalities appears to be rather enigmatic : Is it a matter of length standard ? Is it instead related to some universal constraint in dealing with spatial degrees of freedom ?

6 Less standard example: motion of a particle on the circle

Quantization of the motion of a particle on the circle (like the quantization of polar coordinates in the plane) is an old question with so far unsatisfying answers. A large literature exists concerning this subject, more specifically devoted to the problem of angular localization and related Heisenberg inequalities, see for instance [28].

Let us apply our scheme of coherent state quantization to this particular problem. We just have to follow the steps listed below:

6.1 Coherent states on the circle

- The observation set X is the cylinder $S^1 \times \mathbb{R} = \{x \equiv (\beta, J), \,|\, 0 \leq \beta < 2\pi,\ J \in \mathbb{R}\}$, *i.e.* the phase space of a particle moving on the circle.
- The real J and β are canonically conjugate variables and $dJ\, d\beta$ is the measure invariant with respect to canonical transformations.
- For the measure on X, we choose the invariant measure (up to a factor) : $\mu(dx) = \frac{1}{2\pi}\, dJ\, d\beta$.
- The functions $\phi_n(x)$ forming the orthonormal system needed to construct coherent states are suitably weighted Fourier exponentials:

$$\phi_n(x) = \left(\frac{\epsilon}{\pi}\right)^{1/4} e^{-\frac{\epsilon}{2}(J-n)^2}\, e^{in\beta},\ n \in \mathbb{Z}, \tag{60}$$

  where $\epsilon > 0$ can be arbitrarily small. This parameter could be viewed as the analogue of the Planck constant. Actually it represents a regularization. Notice that the continuous distribution $x \mapsto |\phi_n(x)|^2$ is the normal law centered at n (for the momentum variable J).
- The normalization factor

$$\mathcal{N}(x) \equiv \mathcal{N}(J) = \sqrt{\frac{\epsilon}{\pi}} \sum_{n\in\mathbb{Z}} e^{-\epsilon(J-n)^2} < \infty \tag{61}$$

  is a periodic train of normalized Gaussians and is proportional to an elliptic Theta function. Applying the Poisson summation yields the alternative form :

$$\mathcal{N}(J) = \sum_{n\in\mathbb{Z}} e^{2\pi inJ}\, e^{-\frac{\pi^2}{\epsilon}n^2}. \tag{62}$$

  From this formula we easily prove that $\lim_{\epsilon\to 0}\mathcal{N}(J) = 1$.
- Coherent states read

$$|J, \beta\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \left(\frac{\epsilon}{\pi}\right)^{1/4} \sum_{n\in\mathbb{Z}} e^{-\frac{\epsilon}{2}(J-n)^2}\, e^{-in\beta}|n\rangle, \tag{63}$$

where the states $|n\rangle$'s, in one-to-one correspondence with the ϕ_n's, form an orthonormal basis of some separable Hilbert space $\mathcal{H}$. For instance, they can be considered as Fourier exponentials $e^{in\beta}$ forming the orthonormal basis of the Hilbert space $L^2(S^1) \simeq \mathcal{H}$. They are the *spatial modes* in this representation.

The coherent states (63) have been proposed by De Bièvre-González (1992-93) [29], González-Del Olmo (1998) [31], Kowalski-Rembieliński-Papaloucas (1996) [30]. For recents developments and related discussions, see [32, 33, 34, 35].

The quantization of the observation set is hence achieved by selecting in the (modified) Hilbert space $L^2(S^1 \times \mathbb{R}, \sqrt{\frac{\epsilon}{\pi}}\,\frac{1}{2\pi} e^{-\epsilon J^2}\, dJ\, d\beta)$ all Laurent series in the complex variable $z = e^{\epsilon J - i\beta}$, and this is the choice of polarization [36] leading to our quantization.

6.2 Quantization of classical observables

- By virtue of (35) and (39), the quantum operator (acting on $\mathcal{H}$) associated to the classical observable $f(x)$ is obtained by

$$A_f := \int_X f(x)|x\rangle\langle x|\,\mathcal{N}(x)\mu(dx). \tag{64}$$

- For the most basic one, associated to the classical observable J, this yields

$$A_J = \int_X \mu(dx)\mathcal{N}(J)\, J\, |J,\beta\rangle\langle J,\beta| = \sum_{n\in\mathbb{Z}} n\, |n\rangle\langle n|, \tag{65}$$

  and this is nothing but the angular momentum operator, which reads in angular position representation (Fourier series): $A_J = -i\frac{\partial}{\partial\beta}$.

- For an arbitrary function $f(\beta)$, we have

$$A_{f(\beta)} = \int_X \mu(dx)\mathcal{N}(J) f(\beta)\, |J,\beta\rangle\langle J,\beta| \tag{66}$$

$$= \sum_{n,n'\in\mathbb{Z}} e^{-\frac{\epsilon}{4}(n-n')^2}\, c_{n-n'}(f)|n\rangle\langle n'|, \tag{67}$$

  where $c_n(f)$ is the nth Fourier coefficient of f. In particular, we have for the

– operator "angle" :

$$A_\beta = \pi \mathbb{I}_{\mathcal{H}} + \sum_{n \neq n'} i \frac{e^{-\frac{\epsilon}{4}(n-n')^2}}{n-n'} |n\rangle\langle n'|, \tag{68}$$

– operator "Fourier fundamental harmonic" :

$$A_{e^{i\beta}} = e^{-\frac{\epsilon}{4}} \sum_n |n+1\rangle\langle n|. \tag{69}$$

- In the isomorphic realisation of $\mathcal{H}$ in which the kets $|n\rangle$ are the Fourier exponentials $e^{i\,n\beta}$: $A_{e^{i\beta}}$ is multiplication operator by $e^{i\beta}$ up to the factor $e^{-\frac{\epsilon}{4}}$ (which is arbitrarily close to 1).

6.3 Did you say *canonical* ?

- The "canonical" commutation rule

$$[A_J, A_{e^{i\beta}}] = A_{e^{i\beta}}$$

is canonical in the sense that it is in exact correspondence with the classical Poisson bracket

$$\left\{J, e^{i\beta}\right\} = ie^{i\beta}$$

It is actually the only non trivial commutator having this exact correspondence [44].

- There could be interpretational difficulties with commutators of the type :

$$[A_J, A_{f(\beta)}] = \sum_{n,n'} (n-n') e^{-\frac{\epsilon}{4}(n-n')^2} c_{n-n'}(f)\, |n\rangle\langle n'|,$$

- in particular for the angle operator:

$$[A_J, A_\beta] = i \sum_{n \neq n'} e^{-\frac{\epsilon}{4}(n-n')^2} |n\rangle\langle n'|, \tag{70}$$

to be compared with the classical $\{J, \beta\} = 1$!

Actually, these difficulties are only apparent ones and are due to the discontinuity of the 2π-periodic function $B(\beta)$ which is equal to β on $[0, 2\pi)$. They can be circumvented if we examine, for instance the behaviour of the corresponding lower symbols at the limit $\epsilon \to 0$. For the angle operator,

$$\begin{aligned}\langle J_0, \beta_0 | A_\beta | J_0, \beta_0 \rangle &= \pi + \frac{1}{2}\left(1 + \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)}\right) \sum_{n \neq 0} i \, \frac{e^{-\frac{\epsilon}{2}n^2 + in\beta_0}}{n} \\ &\underset{\epsilon \to 0}{\sim} \pi + \sum_{n \neq 0} i \, \frac{e^{in\beta_0}}{n}, \end{aligned} \tag{71}$$

where we recognize at the limit the Fourier series of $B(\beta_0)$. For the commutator,

$$\begin{aligned}\langle J_0, \beta_0 | [A_J, A_\beta] | J_0, \beta_0 \rangle &= \frac{1}{2}\left(1 + \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)}\right) \left(-i + \sum_{n \in \mathbb{Z}} i e^{-\frac{\epsilon}{2}n^2 + in\beta_0}\right) \\ &\underset{\epsilon \to 0}{\sim} -i + i \sum_n \delta(\beta_0 - 2\pi n). \end{aligned} \tag{72}$$

So we (almost) recover the canonical commutation rule except for the singularity at the origin mod 2π.

7 From the motion of the circle to the motion on 1+1-de Sitter space-time

The material of the previous section is now used to describe the quantum motion of a massive particle on a 1+1-de Sitter background, which means a one-sheeted hyperboloid embedded in a 2+1-Minkowski space. Here, we just summarize the content of Reference [37]. The phase space X is also a one-sheeted hyperboloid:

$$J_1^2 + J_2^2 - J_0^2 = \kappa^2 > 0, \tag{73}$$

with (local) canonical coordinates (J, β), as for the motion on the circle. Phase space coordinates are now viewed as basic classical observables,

$$J_0 = J, \;\; J_1 = J\cos\beta - \kappa \sin\beta, \;\; J_2 = J \sin\beta + \kappa\cos\beta, \tag{74}$$

and obey the Poisson bracket relations

$$\{J_0, J_1\} = -J_2,\ \{J_0, J_2\} = J_1,\ \{J_1, J_2\} = J_0. \tag{75}$$

They are, as expected, the commutation relations of $so(1,2) \simeq sl(2,\mathbb{R})$, which is the kinematical symmetry algebra of the system. Applying the coherent states quantization (64) at $\epsilon \neq 0$ produces the basic quantum observables:

$$A_{J_0} = \sum_n n|n\rangle\langle n|, \tag{76}$$

$$A^{\epsilon}_{J_1} = \frac{1}{2}\, e^{-\frac{\epsilon}{4}} \sum_n (n + \frac{1}{2} + i\kappa)|n+1\rangle\langle n| + \text{ cc}, \tag{77}$$

$$A^{\epsilon}_{J_2} = \frac{1}{2i}\, e^{-\frac{\epsilon}{4}} \sum_n (n + \frac{1}{2} + i\kappa)|n+1\rangle\langle n| - \text{ cc}. \tag{78}$$

The quantization is asymptotically exact for these basic observables since

$$[A_{J_0}, A^{\epsilon}_{J_1}] = iA^{\epsilon}_{J_2},\ [A_{J_0}, A^{\epsilon}_{J_2}] = -iA^{\epsilon}_{J_1},\ [A^{\epsilon}_{J_1}, A^{\epsilon}_{J_2}] = -ie^{-\frac{\epsilon}{4}} A_{J_0}. \tag{79}$$

Moreover, the quadratic operator

$$C^{\epsilon} = (A^{\epsilon}_{J_1})^2 + (A^{\epsilon}_{J_2})^2 - (A_{J_0})^2 \sum_n (e^{-\frac{\epsilon}{4}}(n^2+\kappa^2+\frac{1}{4}) - n^2)|n\rangle\langle n|, \tag{80}$$

admits the limit $C^{\epsilon} \underset{\epsilon\to 0}{\sim} (\kappa^2+\frac{1}{4})\mathbb{I}$. Hence we have produced a coherent states quantization which leads asymptotically to the principal series of $SO_0(1,2)$.

8 Polymer quantization of the motion on the line

The content and terminology of this section have been mainly inspired by the reading of Reference [38] in which is described a toy model of quantum geometry. The game is to rebuild a "shadow" Schrödinger quantum mechanics on all possible discretizations of the real line. A simple way to do this is to adapt the previous frame quantization of the motion on the circle to an arbitrary discretization of the line. Actually, we shall deal with the same phase space as the one for the motion of the particle on the line, *i.e.* the plane, but provided with a different measure, in order to get an arbitrarily discretized quantum position. Like in Section 6, we just enumerate the successive steps of our quantization procedure applied to this specific situation.

- The observation set X is the plane $\mathbb{R}^2 = \{x \equiv (q,p)\}$.
- The measure (actually a functional) on X is partly of the "Bohr type":

$$\mu(f) = \int_{-\infty}^{+\infty} dq \lim_{T\to+\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dp\, f(q,p). \tag{81}$$

- We choose as an orthonormal system of functions $\phi_n(x)$ suitably weighted Fourier exponentials associated to a discrete subset $\gamma = \{a_n\}$ of the real line:

$$\phi_n(x) = \left(\frac{\epsilon}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\epsilon}{2}(q-a_n)^2}\, e^{-ia_n p}. \tag{82}$$

  Notice again that the continuous distribution $x \mapsto |\phi_n(x)|^2$ is the normal law centered at a_n for the position variable q.
- The "graph" (in the Ashtekar language) γ is supposed to be uniformly discrete (there exists a non zero minimal distance between successive elements) in such a way that the aperiodic train of normalized gaussians or "generalized" Theta function

$$\mathcal{N}(x) \equiv \mathcal{N}(q) = \sqrt{\frac{\epsilon}{\pi}} \sum_n e^{-\epsilon(q-a_n)^2} \tag{83}$$

  converges. Poisson summation formulas can exist, depending on the structure of the graph γ [39].
- Accordingly, coherent states read as

$$|x\rangle = |q,p\rangle = \frac{1}{\sqrt{\mathcal{N}(q)}} \sum_n \phi_n^\star(x)|\phi_n\rangle. \tag{84}$$

- Quantum operators acting on $\mathcal{H}$ are yielded by using

$$A_f := \int_X f(x)|x\rangle\langle x|\,\mathcal{N}(x)\mu(dx) \tag{85}$$

What are the issues of such a quantization scheme in terms of elementary observables like position and momentum. Concerning position, the followed algorithm illustrates well the side "spanish inn" of our quantization procedure (and actually of <u>any</u> quantization procedure), since the outcome is precisely a quantization of space, as expected from our choice (82). We indeed obtain for the position :

$$\int_X \mu(dx)\mathcal{N}(q)\, q\, |q,p\rangle\langle q,p| = \sum_n a_n\, |\phi_n\rangle\langle\phi_n|. \tag{86}$$

Hence, the graph γ <u>is</u> the "quantization" of space when the latter is viewed from the point of view of coherent states precisely based on γ. On the other hand, this predetermination of the accessible space on the quantum level has dramatic consequences on the quantum momentum. Indeed, the latter experiences the following "discrete" catastrophe :

$$\int_X \mu(dx)\mathcal{N}(q)\, p|q,p\rangle\langle q,p|$$

$$= \sum_{n,n'} \lim_{T\to+\infty}\left[\frac{\sin\left((a_n-a_{n'})\frac{T}{2}\right)}{(a_n-a_{n'})} - \frac{2}{T}\frac{\sin\left((a_n-a_{n'})\frac{T}{2}\right)}{(a_n-a_{n'})^2}\right]\times$$
$$\times e^{-\frac{\epsilon}{4}\,(a_n-a_{n'})^2}|\phi_n\rangle\langle\phi_{n'}|, \tag{87}$$

and the matrix elements do not exist for $a_n \neq a_{n'}$ and are ∞ for $a_n = a_{n'}$. Nevertheless, a mean of circumventing the problem is to deal with Fourier exponentials $e^{i\lambda p}$ as classical observables, instead of the mere function p. This choice is naturally justified by our original option of a measure adapted to the almost-periodic structure of the space generated by the functions (82). For a single "frequency" λ we get

$$\int_X \mu(dx)\mathcal{N}(q)\, e^{i\lambda p}|q,p\rangle\langle q,p| = \sum_{n,n'} e^{-\frac{\epsilon}{4}\,(a_n-a_{n'})^2}\,\delta_{\lambda,a_{n'}-a_n}|\phi_n\rangle\langle\phi_{n'}|. \tag{88}$$

The generalization to superpositions of Fourier exponentials $e^{i\lambda p}$ is straightforward. In particular, one can define discretized versions of the momentum by considering CS quantized versions of finite differences of Fourier exponentials

$$\frac{1}{i}\frac{e^{i\lambda' p} - e^{i\lambda p}}{\lambda' - \lambda}, \tag{89}$$

in which "allowed" frequencies should belong to the set of "interpositions" $\gamma - \gamma'$ in the graph γ.

9 Conclusion

Various quantization problems can be considered by following the approach described in the present paper. The higher-dimensional generalization (with space of quantum states of infinite or finite dimension) of the canonical case to the phase-space $\mathbb{R}^{2n}$ is straightforward and brings nothing new. On the other hand, we can apply the Kowalski-Rembielinski coherent states [40, 41] on the sphere and their generalisations given in [42] to the quantization of the motion on the sphere S^n for which the phase-space is topologically $\mathbb{R}^n \times S^n$, or equivalently to quantization of the motion of a test particle in a $n+1$-dimensional de Sitter space with symmetry group $SO(1, n+1)$. As a byproduct, we thus obtain the principal series of this group.

Also, we have applied the method to the sphere S^2 considered as an observation set X in order to recover in a rather straightforward way the main features of the so-called fuzzy sphere [8, 43]. Developments in direction of quantum geometry are also in progress.

Let us now attempt to propose in this conclusion, on an elementary level, some hints for understanding better the probabilistic duality lying at the heart of our quantization procedure. Let X be an observation set equipped with a measure ν and let a real-valued function $X \ni x \mapsto a(x)$ have the status of "observable". This means that there exists an experimental device, Δ_a, giving access to a set or "spectrum" of numerical outcomes $\Sigma_a = \{a_j,\ j \in \mathcal{J}\} \subset \mathbb{R}$, commonly interpreted as the set of all possible measured values of $a(x)$. To set Σ_a are attached two probability distributions defined by the set of functions $\Pi_a = \{p_j(x),\ j \in \mathcal{J}\}$:

1. a family of probability distributions on the set $\mathcal{J}$, $\mathcal{J} \ni j \mapsto p_j(x)$, $\sum_{j\in\mathcal{J}} p_j(x) = 1$, indexed by the observation set X. This probability encodes what is precisely expected for this pair (observable $a(x)$, device Δ_a).

2. a family of probability distributions on the measure set (X, ν), $X \ni x \mapsto p_j(x)$, $\int_X p_j(x)\, \nu(dx) = 1$, indexed by the set $\mathcal{J}$.

Now, the exclusive character of the possible outcomes a_j of the measurement of $a(x)$ implies the existence of a set of "conjugate" functions $X \ni x \mapsto \alpha_j(x)$, $j \in \mathcal{J}$, playing the role of phases, and making the set of complex-valued functions

$$\phi_j(x) \stackrel{\text{def}}{=} \sqrt{p_j(x)}\, e^{i\alpha_j(x)} \tag{90}$$

an orthonormal set in the Hilbert space $L^2(X, \nu)$,

$$\int_X \phi_j^*(x)\phi_{j'}(x)\, \nu(dx) = \delta_{jj'}.$$

There follows the existence of the frame in the Hilbertian closure $\mathcal{H}$ of the linear span of the ϕ_j's :

$$X \ni x \mapsto |x\rangle = \sum_{j\in\mathcal{J}} \phi_j^*(x)\, |\phi_j\rangle, \;\; \langle x|x\rangle = 1, \;\; \int_X |x\rangle\langle x|\, \nu(dx) = \mathbb{I}_{\mathcal{H}} \tag{91}$$

A consistency condition has to be satisfied together with this material: it follows from the quantization scheme resulting from the frame (91).

Frame consistency condition

The frame quantization of the observable $a(x)$ produces an essentially self-adjoint operator A in $\mathcal{H}$ which is diagonal in the basis $\{\phi_j,\; j \in \mathcal{J}\}$, with spectrum precisely equal to Σ_a :

$$A \stackrel{\text{def}}{=} \int_X a(x)\, |x\rangle\langle x|\, \nu(dx) = \sum_{j\in\mathcal{J}} a_j |\phi_j\rangle\langle\phi_j|. \tag{92}$$

Once this condition verified, one can start the frame quantization of other observables, for instance the quantization of the conjugate observables $\alpha_j(x)$, and check whether the observational or experimental consequences or constraints due to this mathematical formalism are effectively in agreement with our "Reality".

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