

CASIMIR ENERGIES FOR SINGLE CAVITIES

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Abstract

Old and recent massless scalar field Casimir energy calculations for cavities of several different shapes are discussed. We also attempted to examine the resulting forces.

1 Introduction

The Casimir energy is the vacuum energy due to the quantum fluctuations of the fields in the presence of boundaries. The known two body Casimir interactions are all attractive. In fact the experiments so far are only performed for geometries involving two disconnected boundaries [1, 2].

The signs of the energies in three dimensional cavities however are calculated to be either positive or negative. The well known and the first positive energy example is of the spherical cavity [3]. It is our purpose to have in hand as much examples as possible for three dimensional cavities. We hope that the rapidly progressing nanotechnology can provide experimental verification for single cavities too. At this point we like to stress that in nanostructures the Casimir forces must be taken into account: For a cavity with a typical size r , the Casimir energy (in $\hbar = c = 1$ units) is of the order $10^{-1}/r$ or $10^{-2}/r$. For nanometer distances; that is for $r = 10^{-7}cm$, (with $1eV \cong 0,5 \times 10^5 cm^{-1}$) these energies are $100eV$ or $10eV$, which are of considerable magnitude. In fact it may be that the Casimir force can be used for micro-engineering purposes [4].

In this note we discuss the known and recently obtained Casimir energy calculations for three dimensional cavities. We present only the massless scalar fields results. For geometries with surfaces not parallel or orthogonal to each other one needs to employ the finite groups generated by the reflections from the walls of the cavity on which Dirichlet condition to be imposed. We do not get into calculational details, which can be found at the references given for each subsection of Section II.

2 Single Cavity Results

2.1 A Pyramidal Cavity [5]

First we consider the pyramidal cavity defined by the planes

$$P_1 : z = x, \quad P_2 : y = 0, \quad P_3 : y = z, \quad P_4 : z = a; \quad (1)$$

The above cavity is a specific one; that is the angles are fixed and the size dependence is through the length a only. The region between the planes P_1, P_2, P_3 is the fundamental domain of the octahedral group of order 48 which is generated by the reflections from these planes. The pyramidal cavity is the fundamental domain of the crystallographic octahedral group. This allows us to obtain the Green function in the region satisfying the Dirichlet boundary conditions on the boundary walls [5].

The total Casimir energy for the massless scalar field in this cavity is positive (in $\hbar = c = 1$ units):

$$E_{pyr} \simeq \frac{0,069}{a} > 0. \quad (2)$$

2.2 A Conical Cavity [6]

First we consider the planes

$$P_1 : y = z, \quad P_2 : y = 0, \quad P_3 : y = z; \quad x, y > 0 \quad (3)$$

and impose the Dirichlet boundary condition on P_2 for the massless scalar field Green function

$$K(x, y) |_{\bar{x} \in P_2} = 0. \quad (4)$$

Then we require the identification condition

$$K(x, y) |_{\bar{x} \in P_1} = K(x, y) |_{g\bar{x} \in P_3} . \quad (5)$$

where g is the rotation matrix around the direction $\vec{n} = (1, 1, 1)$ (the intersection of the planes P_1, P_2) by the angle $\frac{2\pi}{3}$. The above boundary conditions define a conical wedge which is the fundamental domain of the tetrahedral group of order 24. Further we consider two more planes

$$P_4 : x = a, \quad P_5 : z = a \quad (6)$$

with Dirichlet boundary conditions imposed on them. The above construction creates a conical cavity with height $h = \frac{2\sqrt{2}}{3}a$ and opening angle $\beta = \arcsin \frac{1}{3}$. The conical cavity appears to be the fundamental

domain of the crystallographic tetrahedral group.

The Casimir energy in the considered cavity is again positive:

$$E_{con} \simeq \frac{0,085}{a} > 0. \quad (7)$$

2.3 Rectangular Prism [7]

The cavity we consider in this subsection is a rectangular prism with square base of edges a and with height b . The sign of the Casimir energy in this cavity exhibits some peculiarities, for it depends on the ratio $\frac{b}{a}$. The value of the energy is

$$E_{rec} \simeq -\frac{0,013}{a} + \frac{(0,011)b}{a^2} \quad \text{for } b > a, \quad (8)$$

and

$$E_{rec} \simeq -\frac{0,013}{b} + \frac{(0,011)a}{b^2} \quad \text{for } a > b. \quad (9)$$

Inspecting the above formulae we see that the energy is positive for $b > \frac{13a}{11} \simeq 1.2a$ or $b < \frac{11a}{13} \simeq 0,8a$; otherwise it is negative.

The force $F_b = -\frac{\partial E}{\partial b}$ in the direction of the height of the prism is negative (attractive) for $b > a$ and positive (repulsive) for $b < a$.

The force $F_a = -\frac{\partial E}{\partial a}$ on the walls perpendicular to the square bases on the other hand is just the opposite; that is, it is positive for $b > a$ and negative for $b < a$.

2.4 Prism with triangular base [8]

The cavity we discuss now is a prism of height b with equilateral triangular cross section of edges a . The Casimir energy for this cavity is given by

$$E_{tri} = \frac{1}{2}(E_{rec} - E_2). \quad (10)$$

Here E_{rec} is the energy for the rectangular prism of the previous subsection. E_2 is the energy for the two dimensional rectangle with

edges b and $\frac{a}{\sqrt{2}}$. We can distinguish three cases

$$E_{tri} \simeq -\frac{0,053}{a} + \frac{(0,029)b}{a^2} \quad \text{for } b > a \quad (11)$$

and

$$E_{tri} \simeq \frac{1}{2} \left(-\frac{0,013}{b} + \frac{(0,011)a}{b^2} + \frac{0,093}{a} - \frac{(0,048)b}{a^2} \right) \quad \text{for } a > b > \frac{a}{\sqrt{2}} \quad (12)$$

and

$$E_{tri} \simeq -\frac{0,039}{b} + \frac{(0,014)a}{b^2} \quad \text{for } b < \frac{a}{\sqrt{2}} \quad (13)$$

When we study the Casimir forces F_b and F_a we observe that for $b > \frac{a}{\sqrt{2}}$ we have $F_b < 0$ and $F_a > 0$; for $b < \frac{a}{\sqrt{2}}$ we have $F_b > 0$ and $F_a < 0$. Like in the case of the rectangular prism, too thin or too thick triangular prisms are not preferred.

3 Discussion

The results we have do not provide any hint for our basic question: What the sign of the Casimir energy depends on?

We have three definitely positive energy values in hand: The pyramidal and conical cavities we briefly discussed and the well known spherical cavity of radius a for which the energy is [3]

$$E_{ball} \simeq \frac{0,046}{a} > 0. \quad (14)$$

If we compare the above positive energy and the energies of pyramidal and conical cavities for equal volumes of these three cavities, we have

$$E_{pyr} \simeq 0,51E_{ball} \quad \text{and} \quad E_{con} \simeq 0,54E_{ball}. \quad (15)$$

That is the energy in the ball is almost twice the energies in other two cavities. To understand this fact we can think of the quantum billiards as an analogy. Sphere which does not have any corner certainly has more space for the motion of the billiards.

The results we have for the rectangular and triangular prisms show that the sign depends very strongly on the shape of the cavities. We may conjecture that the Casimir forces act to make the cavities not too thin or not too thick. However we need more examples in this respect too.

Finally we must say that the sign of the Casimir energy is still very mysterious. Discussing the forces resulting from the Casimir energies appears to be more relevant approach than discussing the sign of the energies.

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