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THE ENERGY-MOMENTUM TENSOR IN CLASSICAL FIELD THEORY

Walter Wyss
Department of Physics
University of Colorado
Boulder, CO 80309

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My friend, Asim Barut, was always interested in classical field theory and in particular in the role that a divergence term plays in a lagrangian. This paper addresses this question via the energy-momentum tensor.

Abstract

We consider the inhomogeneous Lorentz-group as the fundamental symmetry group of all of physics. For Lorentz-invariant actions in classical Lagrange field theories we construct the energy tensor and the energy-momentum tensor. These are intimately related to the generators of the inhomogeneous Lorentz group. With respect to additional symmetry groups of the action, only the physical conservation laws are invariant.

1 Introduction

The concept of the energy-momentum tensor in classical field theory has a long history, especially in Einstein's Theory of Gravity. We consider the inhomogenous Lorentz-group as the fundamental symmetry group of all physics. A physical Lagrange action is then Lorentz invariant. Translation invariance leads to an energy tensor, and Lorentz invariance to an energy-momentum tensor. These tensors are Lorentz-covariant. The energy-momentum tensor is symmetric whereas the energy tensor in general is not symmetric. Any additional symmetry of the action will be treated separately from Lorentz-invariance. In this paper we construct the energy tensor and the energy-momentum tensor for the following action

$$A = \int dx L_0 + \int dx \partial_\mu B^\mu,$$

where L_0 and B^μ depend on the field and its first derivatives. The influence of additional coordinate symmetry of the action will be investigated.

2 The Variational Principle

Let $\eta_{\mu v}$ denote the Lorentz metric tensor with signature $(+,-,-,-)$ and $\eta^{\mu v}$ its inverse. Raising indices is performed with $\eta^{\mu v}$ and lowering indices with $\eta_{\mu v}$.

$\phi = (\phi_\alpha)$ stands for a multicomponent field. The Lorentz 4-volume element is represented by dx. We also use the abbreviations $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\phi_\mu \equiv \partial_\mu \phi$.

We now consider the following action

$$A = \int L\, dx, \tag{1}$$

where

$$L = L_0 + \partial_\mu B^\mu, \tag{2}$$

and

$$L_0 = L_0 \left(\phi, \phi_\mu\right), \tag{3}$$

$$B^\mu = B^\mu \left(\phi, \phi_v\right), \tag{4}$$

i.e. L_0 and B^μ depend on the field ϕ and its first derivatives only.

According to the calculus of variation (Appendix) we get the following relations

$$\delta A = \int dx \delta_* L + \int \partial_\mu \left[L \delta x^\mu \right] \cdot dx, \tag{5}$$

$$\delta A = \int dx \delta_* L_0 + \int \partial_\mu \left[L_0 \delta x^\mu + \left(\partial_\alpha B^\alpha \right) \delta x^\mu + \delta_* B^\mu \right] dx. \tag{6}$$

Definition 1. *1.*

$$H^\mu \equiv \frac{\partial L_0}{\partial \phi_\mu}. \tag{7}$$

2. Euler derivative

$$\varepsilon \left(\phi \right) L_0 = \frac{\partial L_0}{\partial \phi} - \partial_\mu \frac{\partial L_0}{\partial \phi_\mu}. \tag{8}$$

3.

$$G \equiv \varepsilon \left(\phi \right) L_0. \tag{9}$$

Then our variational principle reads

$$\begin{aligned} \delta A = & \int dx \left[\frac{\partial L_0}{\partial \phi} \delta_* \phi + \frac{\partial L_0}{\partial \phi_\mu} \delta_* \phi_\mu \right] \\ & + \int \partial_\mu \left[L_0 \delta x^\mu + \left(\partial_\alpha B^\alpha \right) \delta x^\mu + \delta_* B^\mu \right] dx \\ & = \int G \delta_* \phi \cdot dx + \int \partial_\mu \left[L_0 \delta x^\mu + H^\mu \delta_* \phi + \left(\partial_\alpha B^\alpha \right) \delta x^\mu + \delta_* B^\mu \right] dx. \end{aligned} \tag{10}$$

From the relation

$$\delta_* = \delta - \delta x^\sigma \partial_\sigma,$$

we then get

$$\begin{aligned} \delta A = & \int G \delta \phi dx - \int G \phi_\sigma \delta x^\sigma dx \\ & + \int \partial_\mu \left[L_0 \delta x^\mu - H^\mu \phi_\sigma \delta x^\sigma + H^\mu \delta \phi + \left(\partial_\alpha B^\alpha \right) \delta x^\mu + \delta_* B^\mu \right] dx. \end{aligned} \tag{11}$$

Definition 2. *Energy tensor*

$$E_\sigma^\mu \equiv H^\mu \phi_\sigma - \delta_\sigma^\mu L_0. \tag{12}$$

Using this energy tensor and the Appendix the variation of the action becomes

$$\begin{aligned} \delta A = & \int G\delta\phi dx - \int G\phi_\sigma \delta x^\sigma \cdot dx \\ & + \int \partial_\mu \left[-E_\sigma^\mu \delta x^\sigma + H^\mu \delta\phi \right. \\ & \left. + B^\mu \left(\partial_\alpha \delta x^\alpha\right) - B^\alpha \left(\partial_\alpha \delta x^\mu\right) + \delta B^\mu \right] dx. \end{aligned} \tag{13}$$

3 Coordinate Variations

We assume now that the field ϕ transforms according to a representation of the inhomogeneous Lorentz group. Then

$$\delta\phi = -S_\lambda^\beta \left(\phi\right) \partial_\beta \delta x^\lambda, \tag{14}$$

and the variational principle reads

$$\begin{aligned} \delta A = & \int dx G \left\{ -\phi_\sigma \delta x^\sigma - S_\lambda^\beta \partial_\beta \delta x^\lambda \right\} + \\ & \int \partial_\mu \left[-E_\sigma^\mu \delta x^\sigma - H^\mu S_\lambda^\beta \partial_\beta \delta x^\lambda + B^\mu \partial_\alpha \delta x^\alpha - B^\alpha \partial_\alpha \delta x^\mu \right. \\ & - \left\{ \frac{\partial B^\mu}{\partial \phi} S_\lambda^\beta + \frac{\partial B^\mu}{\partial \phi_\beta} \phi_\lambda + \frac{\partial B^\mu}{\partial \phi_\alpha} \partial_\alpha S_\lambda^\beta \right\} \partial_\beta \delta x^\lambda \\ & \left. - \frac{\partial B^\mu}{\partial \phi_\alpha} S_\lambda^\beta \partial_\alpha \partial_\beta \delta x^\lambda \right] dx. \end{aligned} \tag{15}$$

Definition 3. *1.*

$$P_\lambda^{\mu\beta} \equiv \delta_\lambda^\mu B^\beta - \delta_\lambda^\beta B^\mu + \frac{\partial B^\mu}{\partial \phi} S_\lambda^\beta + \frac{\partial B^\mu}{\partial \phi_\beta} \phi_\lambda + \frac{\partial B^\mu}{\partial \phi_\alpha} \partial_\alpha S_\lambda^\beta. \tag{16}$$

2.

$$K_\lambda^{\mu\alpha} \equiv H^\mu S_\lambda^\alpha + P_\lambda^{\mu\alpha}. \tag{17}$$

The variational principle reads

$$\delta A = \int dx \left\{ -G\phi_\sigma \delta x^\sigma - G S^\beta_\lambda \partial_\beta \delta x^\lambda \right\} + \int \partial_\mu \left[-E^\mu_\sigma \delta x^\sigma - K^{\mu\beta}_\lambda \partial_\beta \delta x^\lambda - \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta_\lambda \partial_\alpha \partial_\beta \delta x^\lambda \right] dx, \tag{18}$$

or

$$\delta A = \int dx \left\{ \partial_\beta \left(G S^\beta_\lambda \right) - G\phi_\lambda \right\} \delta x^\lambda + \int \partial_\mu \left[-\left\{ E^\mu_\sigma + G S^\mu_\sigma \right\} \delta x^\sigma - K^{\mu\beta}_\lambda \partial_\beta \delta x^\lambda - \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta_\lambda \partial_\alpha \partial_\beta \delta x^\lambda \right] dx. \tag{19}$$

4 Lorentz-invariance of the action

If the action is Lorentz-invariant then $\delta A = 0$ for infinitesimal Lorentz transformations

$$\delta x^\mu = \omega^\mu_v x^v + \delta a^\mu, \tag{20}$$

where δa^μ represents an infinitesimal translation and $\omega_{\mu v} = -\omega_{v\mu}$ an infinitesimal proper Lorentz transformation

a) Translation invariant action

Because of $\partial_\beta \delta x^\mu = 0$ and (18) we get

$$\partial_\mu E^\mu_\sigma + G\phi_\sigma = 0. \tag{21}$$

b) Lorentz invariant action

Because of $\delta x^\lambda = \omega^\lambda_\sigma x^\sigma$, $\partial_\beta \delta x^\lambda = \omega^\lambda_\beta$, $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, $\partial_\alpha \partial_\beta \delta x^\lambda = 0$ and (18) we get

$$-G\phi_\sigma \omega^\sigma_\alpha x^\alpha - G S^\beta_\lambda \omega^\lambda_\beta + \partial_\mu \left[-E^\mu_\sigma \omega^\sigma_\alpha x^\alpha - K^{\mu\beta}_\lambda \omega^\lambda_\beta \right] = 0, \tag{22}$$

$\forall \omega^\lambda_\beta$ with $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$.

Definition 4. *1.*

$$E^{\mu\alpha} = \eta^{\alpha\sigma} E^\mu_\sigma. \tag{23}$$

2.

$$K^{\mu\alpha\beta} = \eta^{\lambda\beta} K^{\mu\alpha}_{\lambda}. \tag{24}$$

3.

$$F^{\mu\alpha\beta} \equiv \frac{1}{2}\left[K^{\mu\alpha\beta} - K^{\mu\beta\alpha}\right]. \tag{25}$$

Then

$$F^{\mu\alpha\beta} = -F^{\mu\beta\alpha}, \tag{26}$$

and from (22) we get

$$-G\eta^{\lambda\alpha}\left[x^{\beta}\phi_{\lambda} + S^{\beta}_{\lambda}\right]\omega_{\alpha\beta} + \partial_{\mu}\left[-E^{\mu\alpha}x^{\beta} + F^{\mu\alpha\beta}\right]\omega_{\alpha\beta} = 0. \tag{27}$$

Definition 5. *Angular momentum tensor*

$$M^{\mu\alpha\beta} \equiv \frac{1}{2}\left[E^{\mu\alpha}x^{\beta} - E^{\mu\beta}x^{\alpha}\right] - F^{\mu\alpha\beta}. \tag{28}$$

Then (27) reads

$$\partial_{\mu}M^{\mu\alpha\beta} + \frac{1}{2}G\left[\eta^{\lambda\alpha}x^{\beta} - \eta^{\lambda\beta}x^{\alpha}\right]\phi_{\lambda} + \frac{1}{2}G\left[S^{\beta\alpha} - S^{\alpha\beta}\right] = 0, \tag{29}$$

or equivalently

$$\begin{aligned}\partial_{\mu}F^{\mu\alpha\beta} =& \frac{1}{2}\left[E^{\beta\alpha} - E^{\alpha\beta}\right] + \frac{1}{2}\left[\left(\partial_{\mu}E^{\mu\alpha}\right)x^{\beta} - \left(\partial_{\mu}E^{\mu\beta}\right)x^{\alpha}\right] \\ &+ \frac{1}{2}G\left[\phi_{\lambda}\left\{\eta^{\lambda\alpha}x^{\beta} - \eta^{\lambda\beta}x^{\alpha}\right\} + S^{\beta\alpha} - S^{\alpha\beta}\right].\end{aligned} \tag{30}$$

Translation and Lorentz-invariant action

Combining the above cases a) and b) we get

$$\partial_{\mu}E^{\mu}_{\sigma} = -G\phi_{\sigma}, \tag{31}$$

$$\partial_{\mu}F^{\mu\alpha\beta} = \frac{1}{2}\left[E^{\beta\alpha} - E^{\alpha\beta}\right] + \frac{1}{2}G\left[S^{\beta\alpha} - S^{\alpha\beta}\right]. \tag{32}$$

Definition 6. *1.*

$$\Theta^{\mu\alpha\beta} \equiv F^{\mu\alpha\beta} + F^{\alpha\mu\beta} + F^{\beta\mu\alpha}, \tag{33}$$

2. Energy-momentum tensor

$$T^{\mu\alpha} \equiv E^{\mu\alpha} + GS^{\mu\alpha} + \partial_\beta \Theta^{\mu\alpha\beta}. \tag{34}$$

$\Theta^{\mu\alpha\beta}$ has the symmetry property

$$\Theta^{\mu\alpha\beta} = -\Theta^{\beta\alpha\mu}. \tag{35}$$

The energy-momentum tensor then can be written as

$$T^{\mu\alpha} = \frac{1}{2}\left(E^{\mu\alpha} + E^{\alpha\mu}\right) + \frac{1}{2}G\left(S^{\mu\alpha} + S^{\alpha\mu}\right) + \partial_\beta\left[F^{\mu\alpha\beta} + F^{\alpha\mu\beta}\right]. \tag{36}$$

It has the following properties

1. $$T^{\mu\alpha} = T^{\alpha\mu}. \tag{37}$$

2. $$\partial_\mu T^{\mu\alpha} = -G\phi_\lambda \eta^{\lambda\alpha} + \partial_\mu\left(GS^{\mu\alpha}\right). \tag{38}$$

The angular momentum tensor (28) can now be expressed, using the energy-momentum tensor, as

$$\begin{aligned} M^{\mu\alpha\beta} =& \frac{1}{2}\left(T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha\right) + \frac{1}{2}\partial_\lambda\left(\Theta^{\lambda\alpha\mu}x^\beta - \Theta^{\lambda\beta\mu}x^\alpha\right) \\ & -\frac{1}{2}G\left(S^{\mu\alpha}x^\beta - S^{\mu\beta}x^\alpha\right). \end{aligned}$$

Remarks

1) The energy tensor $E^{\mu\alpha}$ is in general not symmetric. It is related to translation invariance.
2) The energy-momentum tensor $T^{\mu\alpha}$ is symmetric. It is related to proper Lorentz invariance.

5 Additional Symmetry of the Action

Here we assume that the action is invariant under a larger coordinate transformation group than the inhomogeneous Lorentz-group.

Translation and affine invariant action

Translation invariance implies from (21)

$$\partial_\mu E^\mu_\sigma = -G\phi_\sigma. \tag{39}$$

Infinitesimal affine transformations are given by

$$\delta x^\lambda = A^\lambda_\alpha x^\alpha, \tag{40}$$

where $A^\alpha{}_\lambda$ are infinitesimal constants with no symmetry. Then we get from (18) and (39)

$$\partial_\mu K^{\mu\alpha\beta} = -E^{\alpha\beta} - GS^{\alpha\beta}. \tag{41}$$

This relation is more restrictive on the auxiliary quantity $K^{\mu\alpha\beta}$ than Eq. (32). The energy-momentum tensor however is still given by

$$T^{\mu\alpha} = \frac{1}{2}\left(E^{\mu\alpha} + E^{\alpha\mu}\right) + \frac{1}{2}G\left(S^{\mu\alpha} + S^{\alpha\mu}\right) \tag{42}$$

$$+\partial_\beta\left(F^{\mu\alpha\beta} + F^{\alpha\mu\beta}\right). \tag{43}$$

General covariant action

Here the infinitesimal transformations δx^α are arbitrary. From translation invariance we have

$$\partial^\mu E^\mu_\sigma = -G\phi_\sigma, \tag{44}$$

and from affine invariance

$$\partial_\mu K^{\mu\alpha\beta} = -E^{\alpha\beta} - GS^{\alpha\beta}. \tag{45}$$

Eq. (18) then reduces to

$$K^{\alpha\beta}_\lambda \partial_\alpha\partial_\beta\delta x^\lambda + \partial_\mu\left\{\frac{\partial B^\mu}{\partial\phi_\alpha}S^\beta_\lambda\right\}\cdot\partial_\alpha\partial_\beta\delta x^\lambda + \frac{\partial B^\mu}{\partial\phi_\alpha}S^\beta_\lambda\partial_\mu\partial_\alpha\partial_\beta\delta x^\lambda = 0. \tag{46}$$

For arbitrary δx^λ this implies

1.

$$K^{\alpha\beta}_\lambda + \partial_\mu\left\{\frac{\partial B^\mu}{\partial\phi_\alpha}S^\beta_\lambda\right\} = -\left[K^{\beta\alpha}_\lambda + \partial_\mu\left\{\frac{\partial B^\mu}{\partial\phi_\beta}S^\alpha_\lambda\right\}\right]. \tag{47}$$

2.

$$\sum_{(\mu,\alpha,\beta)} \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta_\lambda = 0. \tag{48}$$

where (μ, α, β) indicate the sum over all permutations of μ, α, β. From equations (25), (33) we find

$$\begin{aligned} 2\Theta^{\mu\alpha\beta} =& 2K^{\beta\mu\alpha} - \left(K^{\beta\mu\alpha} + K^{\mu\beta\alpha}\right) \\ &+ \left(K^{\mu\alpha\beta} + K^{\alpha\mu\beta}\right) - \left(K^{\alpha\beta\mu} + K^{\beta\alpha\mu}\right), \end{aligned} \tag{49}$$

with

$$K^{\alpha\beta\mu} = \eta^{\mu\lambda} K^{\alpha\beta}_\lambda, \tag{50}$$

and

$$S^{\beta\mu} = \eta^{\mu\lambda} S^\beta_\lambda. \tag{51}$$

From

$$K^{\alpha\beta\lambda} + K^{\beta\alpha\lambda} = -\partial_\mu \left[\frac{\partial B^\mu}{\partial \phi_\alpha} S^{\beta\lambda} + \frac{\partial B^\mu}{\partial \phi_\beta} S^{\alpha\lambda}\right]. \tag{52}$$

we then get

$$\Theta^{\mu\alpha\beta} = K^{\beta\mu\alpha} + \partial_\sigma W^{\sigma,\mu\alpha\beta}, \tag{53}$$

where

$$\begin{aligned} W^{\sigma,\mu\alpha\beta} =& \frac{1}{2}\Big[\frac{\partial B^\sigma}{\partial \phi_\alpha} S^{\beta\mu} + \frac{\partial B^\sigma}{\partial \phi_\beta} S^{\alpha\mu} + \frac{\partial B^\sigma}{\partial \phi_\mu} S^{\beta\alpha} \\ &+ \frac{\partial B^\sigma}{\partial \phi_\beta} S^{\mu\alpha} - \frac{\partial B^\sigma}{\partial \phi_\mu} S^{\alpha\beta} - \frac{\partial B^\sigma}{\partial \phi_\alpha} S^{\mu\beta}\Big]. \end{aligned} \tag{54}$$

The energy momentum tensor (34) becomes

$$T^{\mu\alpha} = E^{\mu\alpha} + GS^{\mu\alpha} + \partial_\beta \left[K^{\beta\mu\alpha} + \partial_\sigma W^{\sigma,\mu\alpha\beta}\right]. \tag{55}$$

Due to (45) we then get

$$T^{\mu\alpha} = \partial_\beta \partial_\sigma W^{\sigma,\mu\alpha\beta}. \tag{56}$$

Remarks

1) General covariance is very restrictive on the form of the Lagrange density; the corresponding energy-momentum tensor is a divergence.
2) The corresponding energy tensor satisfies

$$\partial_\mu E^\mu_\sigma = -G\phi_\sigma.$$

6 Equations of motion and conservation laws

The equations of motion are given by demanding that the action Eq. (1) is stationary, i.e. has a critical point under variations where on the boundary of the integration domain $\delta x^\mu = 0, \delta_* \phi = 0$ and $\delta_* \phi_\mu = 0$.

Then we get from Eq. (10) that the equations of motion are given by

$$G = 0. \tag{57}$$

From Eq. (21) we get the conservation law of the energy tensor

$$\partial_\mu E^\mu_\sigma = 0. \tag{58}$$

From Eq. (29) we get the conservation law for the angular momentum tensor

$$\partial_\mu M^{\mu\alpha\beta} = 0. \tag{59}$$

From Eq. (38) we get the conservation law for the symmetric energy-momentum tensor

$$\partial_\mu T^{\mu\alpha} = 0. \tag{60}$$

7 Examples

1)

$$A = \int dx\, L\,(g, \phi). \tag{61}$$

The fields are a covariant tensor field $g_{\alpha\beta}$ ($g^{\alpha\beta}$ it's inverse) and a scalar field ϕ. The Lagrange density is given by

$$L\,(g, \phi) = \sqrt{g}\,\left[g^{\alpha\beta}\,(\partial_\alpha \phi)\,(\partial_\beta \phi) - m^2 \phi^2\right]. \tag{62}$$

The relevant auxiliary quantities are

$$\begin{aligned} H^{\alpha\beta,\mu} &= \frac{\partial L}{\partial g_{\alpha\beta,\mu}} = 0 \\ H^\mu &= \frac{\partial L}{\partial \phi_\mu} = 2\sqrt{g}\, g^{\mu\alpha} \phi_\alpha \\ G^{\alpha\beta} &= \varepsilon\,(g_{\alpha\beta})\, L = \frac{1}{2} g^{\alpha\beta} L - \sqrt{g}\, g^{\mu\alpha} g^{v\beta} \phi_\mu \phi_v \\ G &= \varepsilon\,(\phi)\, L = -2\sqrt{g}\, m^2 \phi - \partial_\mu \left[2\sqrt{g}\, g^{\mu\alpha} \phi_\alpha\right]. \end{aligned} \tag{63}$$

The energy tensor then becomes

$$E^{\mu}_{\sigma} = 2\sqrt{g}\, g^{\mu\alpha}\phi_{\alpha}\phi_{\sigma} - \delta^{\mu}_{\sigma}L. \tag{64}$$

The representations of the Lorentz group are given as follows
(i) scalar field $\phi : S^{\beta}_{\lambda} = 0$
(ii) covariant tensor field $g_{\alpha\beta}$

$$\left(S^{\beta}_{\lambda}\right)_{\sigma\rho} = \delta^{\beta}_{\sigma} g_{\lambda\rho} + \delta^{\beta}_{\rho} g_{\sigma\lambda}.$$

The other auxiliary quantities are then given by

$$\begin{aligned} P^{\mu\beta}_{\lambda} &= 0, \\ K^{\mu\alpha}_{\lambda} &= 0, \\ F^{\mu\alpha\beta} &= 0, \\ \Theta^{\mu\alpha\beta} &= 0. \end{aligned}$$

The energy-momentum tensor now becomes

$$\begin{aligned} T^{\mu\alpha} =& E^{\mu}_{\sigma}\eta^{\sigma\alpha} + G^{\mu\rho}\eta^{\alpha\lambda}\left(S^{\mu}_{\lambda}\right)_{\sigma\rho} \\ =& \eta^{\alpha\sigma}\left[E^{\mu}_{\sigma} + G^{\lambda\rho}\left(S^{\mu}_{\sigma}\right)_{\lambda\rho}\right] \\ =& \eta^{\alpha\sigma}\left[2\sqrt{g}\, g^{\mu\beta}\phi_{\beta}\phi_{\sigma} - \delta^{\mu}_{\sigma}L + 2G^{\mu\rho}g_{\sigma\rho}\right] \\ T^{\mu\alpha} =& 0. \end{aligned} \tag{65}$$

Because our action is invariant under general coordinate transformations, this result agrees with the general result (55).

2) $A = \int dx \partial_{\mu} B^{\mu}$ where $B^{\mu} = B^{\mu}(\phi, \phi_{\alpha})$. ϕ is a general field. The Lagrange density thus consists of a boundary term only. The relevant auxiliary quantities are

$$
\begin{aligned}
H^{\mu} =& 0, \\
G =& 0, \\
E^{\mu}_{\sigma} =& 0, \\
K^{\mu\alpha}_{\lambda} =& P^{\mu\alpha}_{\lambda}, \\
P^{\mu\alpha}_{\lambda} =& \delta^{\mu}_{\lambda} B^{\alpha} - \delta^{\alpha}_{\lambda} B^{\mu} + \frac{\partial B^{\mu}}{\partial \phi} S^{\alpha}_{\lambda} + \frac{\partial B^{\mu}}{\partial \phi_{\alpha}} \phi_{\lambda} + \frac{\partial B^{\mu}}{\partial \phi_{\beta}} \partial_{\beta} S^{\alpha}_{\lambda}, \\
F^{\mu\alpha\beta} =& \frac{1}{2} \left[P^{\mu\alpha\beta} - P^{\mu\beta\alpha} \right].
\end{aligned}
$$

The energy-momentum tensor then becomes

$$
\begin{aligned}
T^{\mu\alpha} =& \partial_{\beta} \left[F^{\mu\alpha\beta} + F^{\alpha\mu\beta} \right] \\
=& \partial_{\beta} \frac{1}{2} \left[P^{\mu\alpha\beta} + P^{\alpha\mu\beta} - P^{\mu\beta\alpha} - P^{\alpha\beta\mu} \right].
\end{aligned} \tag{66}
$$

We also have from (32) and (38) the relations

$$
\begin{aligned}
\partial_{\mu} F^{\mu\alpha\beta} &= 0, \\
\partial_{\mu} T^{\mu\alpha} &= 0.
\end{aligned}
$$

For this action where the Lagrange density is a boundary term only, we have no equations of motion. The energy tensor vanishes but there is a conserved, nontrivial energy-momentum tensor.

8 Conclusion

We looked at physical systems described by a classical Lagrange action. The Lagrange density depends on the field, its first and its second derivatives but in such a form that the equations of motion (Euler derivative) are partial differential equations of second order. We consider the inhomogeneous Lorentz group as the fundamental symmetry group of all of physics. Translation invariance leads to the energy tensor E^{μ}_{σ}, proper Lorentz invariance to the angular momentum tensor $M^{\mu\alpha\beta}$ and invariance with respect to the combined inhomogeneous Lorentz group leads to the symmetric energy-momentum tensor $T^{\mu\alpha}$. If the equations of motion are satisfied we get physical conservation laws. In case the equations of motion have an additional symmetry besides Lorentz convariance the physical conservation laws also have that symmetry. If the action is invariant under general coordinate transformations the energy-momentum tensor is a divergence. This is the case in Einstein's Theory of Gravity. Applications to electromagnetic interactions and the theory of gravity will be published separately.

9 Appendix - Calculus of Variation

Let x represent a point in Minkowski space. In a local coordinate basis x has the coordinates $\{x^\mu\}$. Let ϕ be a multicomponent field defined on Minkowski space. An infinitesimal variation of ϕ results in the field $\bar{\phi}$. The field variation (local variation) is then defined by

$$(\delta_* \phi)(x) = \bar{\phi}(x) - \phi(x).$$

δ_* commutes with the derivative, i.e.

$$\delta_* \partial_\mu = \partial_\mu \delta_*.$$

An infinitesimal coordinate transformation on x^μ results in new coordinates $\bar{x}^\mu$. The coordinate variation is then defined by

$$\delta x^\mu = \bar{x}^\mu - x^\mu.$$

The total variation of a field ϕ, induced by a coordinate variation, is given by

$$(\delta \phi)(x) = \bar{\phi}(\bar{x}) - \phi(x).$$

We then have the relation

$$\delta_* = \delta - (\delta x^\mu)\, \partial_\mu,$$

as applied to any field. We have the following total variations

$$\delta \phi = -S_\lambda^\beta(\phi)\, \partial_\beta \delta x^\lambda.$$

1) scalar field

$$S_\lambda^\beta = 0.$$

2) covariant vector field

$$\left(S^{\beta}_{\lambda}\left(\phi\right)\right)_{\alpha}=\delta^{\beta}_{\alpha}\phi_{\lambda}.$$

3) covariant tensor field

$$\left(S^{\beta}_{\lambda}\left(\phi\right)\right)_{\sigma\rho}=\delta^{\beta}_{\sigma}\phi_{\lambda\rho}+\delta^{\beta}_{\rho}\phi_{\sigma\lambda}.$$

4) spinor field

$$S^{\beta}_{\lambda}\left(\phi\right)=\frac{1}{8}\left(\gamma^{\beta}\gamma^{\mu}-\gamma^{\mu}\gamma^{\beta}\right)\phi\eta_{\mu\lambda}.$$

The variation of the action is given by

$$\delta A=\delta\int Ldx=\int\left(\delta_{*}L\right)dx+\int\partial_{\mu}\left[L\delta x^{\mu}\right]\cdot dx.$$

For the action

$$A=\int dx\left[L_0+\partial_{\mu}B^{\mu}\right],$$

where $L_0=L_0\left(\phi,\phi_{\mu}\right)$, $B^{\mu}=B^{\mu}\left(\phi,\phi_{\alpha}\right)$, with

$$H^{\mu}\equiv\frac{\partial L_0}{\partial\phi_{\mu}},$$

and the Euler derivative

$$\varepsilon\left(\phi\right)L_0=\frac{\partial L_0}{\partial\phi}-\partial_{\mu}H^{\mu}\equiv G,$$

we have the following statements

Lemma 1.

$$\delta_{*}L_0=G\delta_{*}\phi+\partial_{\mu}\left[H^{\mu}\delta_{*}\phi\right]$$

.

Proof.

$$\begin{aligned}\delta_{*}L_0=&\frac{\partial L_0}{\partial\phi}\delta_{*}\phi+\frac{\partial L_0}{\partial\phi_{\mu}}\delta_{*}\phi_{\mu}\\=&\frac{\partial L_0}{\partial\phi}\delta_{*}\phi+\partial_{\mu}\left[H^{\mu}\delta_{*}\phi\right]-\left(\partial_{\mu}H^{\mu}\right)\delta_{*}\phi\\\delta_{*}L_0=&G\delta_{*}\phi+\partial_{\mu}\left[H^{\mu}\delta_{*}\phi\right].\end{aligned}$$

□

Lemma 2.

$$\int \partial_\mu \left[(\partial_\alpha B^\alpha) \, \delta x^\mu + \delta_* B^\mu \right] dx =$$
$$= \int \partial_\mu \left[B^\mu \left(\partial_\alpha \delta x^\alpha \right) - B^\alpha \left(\partial_\alpha \delta x^\mu \right) + \delta B^\mu \right] dx.$$

Proof.

$$\int \partial_\mu \left[\partial_\alpha \left(B^\alpha \delta x^\mu \right) - B^\alpha \partial_\alpha \delta x^\mu - (\partial_\alpha B^\mu) \, \delta x^\alpha + \delta B^\mu \right] dx$$
$$= \int \partial_\mu \left[\partial_\alpha \left(B^\mu \delta x^\alpha \right) - B^\alpha \partial_\alpha \delta x^\mu - (\partial_\alpha B^\mu) \, \delta x^\alpha + \delta B^\mu \right] dx$$
$$= \int \partial_\mu \left[B^\mu \partial_\alpha \delta x^\alpha - B^\alpha \partial_\alpha \delta x^\mu + \delta B^\mu \right] dx.$$

□

Lemma 3.

$$\delta \phi_\alpha = \partial_\alpha \left(\partial \phi \right) - \phi_\sigma \partial_\alpha \delta x^\sigma$$

.

Proof. From

$$\delta_* = \delta - \delta x^\sigma \partial_\sigma,$$

and

$$\delta_* \partial_\alpha = \partial_\alpha \delta_*,$$

we get

$$\begin{aligned} \delta \phi_\alpha =& \delta_* \phi_\alpha + \delta x^\sigma \phi_{\alpha\sigma} \\ =& \partial_\alpha \left[\delta \phi - \delta x^\sigma \phi_\sigma \right] + \delta x^\sigma \phi_{\alpha\sigma} \\ \delta \phi_\alpha =& \partial_\alpha \left(\delta \phi \right) - \phi_\sigma \partial_\alpha \delta x^\sigma . \end{aligned} \tag{67}$$

□

Lemma 4.

$$\delta B^\mu = - \left[\frac{\partial B^\mu}{\partial \phi} S^\beta_\lambda + \frac{\partial B^\mu}{\partial \phi_\beta} \phi_\lambda + \frac{\partial B^\mu}{\partial \phi_\alpha} \partial_\alpha S^\beta_\lambda \right] \partial_\beta \delta x^\lambda \tag{68}$$
$$- \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta_\lambda \partial_\alpha \partial_\beta \delta x^\lambda . \tag{69}$$

Proof.

$$\begin{aligned}\delta B^\mu =& \frac{\partial B^\mu}{\partial \phi}\delta\phi + \frac{\partial B^\mu}{\partial \phi_\alpha}\delta\phi_\alpha = \frac{\partial B^\mu}{\partial \phi}\delta\phi + \frac{\partial B^\mu}{\partial \phi_\alpha}\left(\partial_\alpha \delta\phi - \phi_\sigma \partial_\alpha \delta x^\sigma\right) \\ =& - \frac{\partial B^\mu}{\partial \phi} S^\beta_\lambda \partial_\beta \delta x^\lambda - \frac{\partial B^\mu}{\partial \phi_\alpha}\phi_\sigma \partial_\alpha \delta x^\sigma - \frac{\partial B^\mu}{\partial \phi_\alpha}\partial_\alpha \left(S^\beta_\lambda \partial_\beta \delta x^\lambda\right).\end{aligned}$$

□

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